# TOWARD AN INTERVAL FUZZY THEORY

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## Abstract

The development of robust and trusted systems requests the use of techniques capable to implement the day-by-day problems. In the implementation of systems that use the approximate reasoning, the fuzzy systems have been used with success. However, there are situations where it doesn't satisfy totally the characteristics of the problem. So, trying to improve the results alternatives have been searched. One of these alternatives is the use of the interval theory to fuzzy systems trying to minimize the implementation errors of the values supplied by the specialist. In this work is described the principal components needed for the development of interval fuzzy systems, that is, the development of fuzzy systems integrated with intervals.

## INTRODUCTION

The development of efficient and qualified computational systems is an arduous task. In the search of this quality and efficiency the researchers of the whole world are seeking techniques more and more specialized to solve the day by day problems.

The fuzzy systems have been used with success in the last years, in problems that involve the approximate reasoning [Zadeh 1965, Bojadzieve 1996, Gottgtroy 1996, Kandel 1996, Kasabov 1996, Nguyen 1999, Oliveira 2000]. However, in the search of improved solutions in those systems many researchers are studying other areas and producing works that associate for instance, the Fuzzy Theory and the Interval Mathematics as in [Kearfott 1996, Rocha 1996, Mukaidono 1999, Kreinovich 1999, Yam 1999, Kreinovich 2000a, Kreinovich 2000b] and many others. In this way, many books, papers and monographs of fuzzy set and fuzzy techniques have been dedicating a chapter on Interval computation as [Bojadzieve 1996, Kearfott 1996, Nguyen 1999].

The objective of that work is to specify the fuzzy systems with the addition of the interval theory in its components. The integration of the interval theory to the fuzzy systems is not a trivial task. This process involves several stages, from the definition of operations and properties, generating a interval fuzzy theory, based on the operations and defined properties in the interval and fuzzy theories, respectively, as exemplified further on with union, intersection and complement properties. Another point that should be considered is the construction of the membership function for an interval fuzzy system, where the notion of intervals will be used producing an interval function. Besides, should be built inference and defuzzification methods to close the specification of the interval fuzzy system.

This paper is divided in the following way: in the next session the interval theory will be introduced, containing the definitions of the interval arithmetic operations and interval functions, then the interval fuzzy theory will be placed, showing a motivation, some properties, the definition of the interval membership function and the principal problems of the development of interval fuzzy systems, and at last the final considerations.

## INTERVAL THEORY

The quality of the result in the scientific computation depends on the knowledge and control of the errors in the data processing. To solve this subject the interval mathematics appeared based on Moore's arithmetic [Moore 1966].

Three types of sources of errors exist in classic numeric computation (which represents real numbers as floating point): the propagation of the error in the initial data, the rounding and the truncation error. The interval mathematics tries to solve this problem that concentrates fundamentally in two aspects:

- The creation of a computational model that expresses the control and the analysis of the errors that happen in the computational process;
- The choice of appropriate programming techniques for development of scientific software that minimize the errors in the results.

The user cannot affirm the accuracy of an estimated answer without the aid of an error analysis that is extensive, costly and not always viable. However, the interval mathematics tries to give support to these problems, using the interval arithmetic defined by Moore [Moore 1966].

### Interval Arithmetic

As well as the theory of the fuzzy sets, the theory of the intervals has an arithmetic well developed, where are defined the main arithmetic operations for intervals, based on the respective real arithmetic operations at the extremes of the intervals [Moore 1966, Oliveira 1997].

**Basic definitions:**

**Interval of Real**

Let R the set of the real numbers, and let $a, b \in R$, for wich $a \leq b$. Then the set $\{ x \in R / a \leq x \leq b\}$ is an interval of real or simply an interval, that will be denoted for: $X = [a, b] = \{ x \in R / a \leq x \leq b\}$. The points of the interval set of Real will be denoted by latin capital letters, like X, Y, Z,...

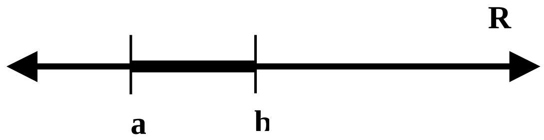


Figure 1: An Interval in the Real Straight line R

Example: The Intervals [2, 3], [5, 5], [-7, -3] and [-6, 4] are examples of some intervals. Observe that the intervals in the way [a, a] represent real numbers, as the interval [5, 5], that represents the real number 5, because the only element of the Interval [5, 5] it is the real number 5. These intervals are called *degenerate intervals*.

The set of all the intervals is defined as being
$IR = \{[a,b] / a,b \in R \text{ e } a \leq b\}$

**Equality**

Let $X = [x_1, x_2]$ and $Y = [y_1, y_2]$ two intervals of IR. Then $X = Y$ if, and only if, $x_1 = y_1$ and $x_2 = y_2$.

As well as the equality, the definition of the interval arithmetic operations can be made through the operations with sets of the real straight line. Those operations are shown below.

**Interval Arithmetic Operations**

Let X = [a, b] and Y = [c, d]. The arithmetic operations on intervals are defined in the ends of their intervals, as is shown below [Moore 1966]

**Sum**:
$X + Y = [a, b] + [c, d] = [a + c, b + d]$

**Pseudo Inverse Addictive**
Let X = [a, b] then:
$- X = [-b, -a]$

**Subtraction**:
$X - Y = X + (-Y) = [a, b] + [-d, -c] = [a - d, b - c]$

**Multiplication**:
$X. Y = [a, b]. [c, d] = [\min\{ac, bc, ad, bd\}, \max\{ac, ad, bc, bd\}]$

**Inverse pseudo Multiplicative**
Let X = [a, b], such that $[0,0] \notin X$ then:
$X-1 = 1/X = [1/b, 1/a]$

**Division**:
$X / Y = X. 1/Y = [a, b]. [1/d, 1/c]$
$= [\min\{a/d, b/d, a/c, b/c\}, \max \{a/d, b/d, a/c, b/c\}]$, if $0 \notin [c, d]$

Can be met the proof and examples of those operations in [Oliveira 1997]. As well as the operations, a lot of properties are also defined on intervals.

**Interval Function**

Let f: X → Y a function.

If X = Dom(f) $\subseteq$ IR e Y = Cod(f) $\subseteq$ IR, X → f(X), then f is an interval function of an interval variable.

Example: f : IR → IR as an interval function.
X → f (X) = [2, 3].X + [4, 5]

## INTERVAL FUZZY THEORY

In the classic logic only exist two possibilities to represent the true values of a proposition: true and false, that are represented respectively in the computer as 1 and 0.

To represent the uncertainties, Zadeh created in 1965 the fuzzy logic, uses membership degrees for the truth-values in the interval [0, 1]. This notation has been used with success in a lot of applications.

However in some situations it is desirable that the values are not points in the interval [0, 1], but more generic values. In [Yam 1999] is given three examples of this situation, but we will just show the first, that treats of a more appropriate representation of membership degree:

The logic fuzzy is based on the interval [0, 1], that describes a specialist's uncertainties through a real number. However if a specialist is uncertain on something, he can be uncertain about your faith in the membership degree of this thing, and is possible that the specialist is not capable to describe his uncertainty degree exactly; that is, a person significantly can distinguish among a membership degree of 0.6 and 0.7, but nobody is capable to describe that your faith of the membership degree is 0.6 and not 0.601. So, a more appropriate representation of membership degrees is not a simple real number, but an interval or a set of possible real numbers.

Worse, a specialist is capable to affirm that exists a membership degree $\pi$ - 3? This type of problem is frequent in systems that treat approaches and rounding. For these problems the ideal is to work with interval computation, to minimize the specialist's error. And besides the system treats of imprecise knowledge? Then it is necessary the use of the Fuzzy Sets Theory. And why not to use the two solutions for the problem?

Some researchers are working with the notion of " Interval Valued Fuzzy Sets (IVFS)", like Turksen, that proposed this theory initially in 1986 [Turksen 1986]; Rocha in [Rocha 1996, 1997a and 1997b] that proposed " Evidence Sets " based on Turksen. And finally in [Kreinovich 2000a, 2000b] the notion of intervals is shown as pairs of fuzzy values, among others.

### Operations and Properties

The relationships between the fuzzy operators and the interval operators should be established. All the fuzzy operators can be naturally extended as intervals [Yam 1999]. For instance, the defined basic operations for fuzzy sets:

Union: A or B = $\{(x, \max(\mu_A(x); \mu_B(x)))\}$
Intersection: A and B = $\{(x, \min((\mu_A(x); \mu_B(x)))\}$
Complement: not A = $\{(x, \mu_{\neg A}(x)) | \mu_{\neg A}(x)=1 - \mu_A(x)\}$
where $x \in U$ (universe in discussion). $\mu_A$ is membership degree to the fuzzy set A. They can be extended as fuzzy interval.

Defining the fuzzy interval set as A= $\{(x, \varphi_A(x)) / x \in R$ and $\varphi_A(x) \in I[0,1]\}$, where $\varphi_A$ is membership degree to the interval fuzzy set A and I[0, 1] is interval set between 0 and 1, with I [0, 1] = $\{ [a, b] \in IR / 0 \le a \le b \le 1\}$. The operations MIN and MAX, the minimum and the maximum for intervals, as:
$MAX([a_1, a_2]; [b_1,b_2]) = [\max(a_1,b_1),\max(a_2,b_2)]$
$MIN([a_1, a_2]; [b_1,b_2]) = [\min(a_1,b_1),\min(a_2,b_2)]$

Let A and B fuzzy interval sets with $\varphi_A(x) = [a_1, a_2]$ and $\varphi_B(x)=[b_1,b_2]$.

Can be defined the same operations for intervals as:
Union: = $\{(x, [\max(a_1,b_1), \max(a_2,b_2)])\}$
Intersection: = $\{(x, [\min(a_1,b_1), \min(a_2,b_2)])\}$
Complement: = $\{(x, [1-a_2, 1-a_1])\}$

### Membership Function

The construction of the membership function is made using intervals in the fuzzy membership function. So, instead of treating membership value as a real number in the image, the membership function defined for the fuzzy interval sets will work with an interval of real numbers in the interval [0, 1], or will be sub-intervals of the interval [0, 1] in the image. In this way, the membership function is a interval function with the image in I [0, 1], with I [0, 1] = $\{ [a, b] \in IR / 0 \le a \le b \le 1\}$. This is, $\varphi_A \in$ I [0, 1].

### The Problems

Because of these relationships the interval methods are used thoroughly in fuzzy applications. Frequently, however, the researchers use old interval techniques, where more advanced techniques could lead for much more efficient and effective results. That disconnection is caused by two reasons: [Kreinovich 2000]

- On a side, many researchers of the fuzzy methods area are not very familiarized with the last tendencies of the interval computation;
- On the other side, the researchers of Intervals are not also familiarized with the problems and methods of the fuzzy techniques.

## FINAL CONSIDERATIONS

The integration of interval theory to systems fuzzy is a subject that is being studied a lot in the scientific community, but it is not totally sedimented yet.

For the specification of the fuzzy interval theory proposed, besides the relationships between the operators and properties and the addition of sub-intervals of the interval [0, 1] to the membership function, is necessary, to finish this initial fuzzy interval study, the description of an fuzzy interval inference as well as of a method of corresponding defuzzification, that will be future works.

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