Intervals as a Categorical Constructor

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Abstract

In this work, from the category sight, we will give a generalized interval theory which makes possible, among other things, to study generic properties of datas which are "intervals" of another datas. In doing so we could obtain some properties which holds for real intervals, complex intervals, interval vectors, interval matrixes, and so on. We introduce a categorical interval constructor on **POSET** *for this purpose and we study the categorical properties that this constructor satisfies in order to define the notion of interval category. We prove also that* **TOP** *and severeal subcategories of* **POSET** *are interval categories.*

1 Introduction

The category theory studies, as primitive concepts, "objects" and "morphisms" between them. The morphisms establish relationships between the objects: any use of the inner structure of the objects is forbidden. It means that every property of the objects must be specified through the properties of the morphisms (existence of particular morphisms, its unicity, some equations which are satisfied by them, etc.). From this point of view, the objects should be considered as an abstract data.

Categories gives an strongly formalized language which is adequated in order to establish abstract properties of mathematical structures.

On the other hand, R. Moore [10, 11] developed an interval mathematic in order to proporcionate a controle of the computational errors resulting of numeric computations involving real numbers. The Moore theory not only comprehend real intervals, but also complex intervals, matrix and array of real and complex intervals.

Thus, it is reasonable to generalize the interval theory, in such a way that include the above kind of intervals and any other possible type of intervals. Since real intervals are defined through a partial order on the real set, we might define intervals on any partially order set.

In this work we propose a general interval theory, based on the **POSET** category and an interval constructor introduced in [4]. We study the categorical properties that this constructor satisfies in order to define the notion of interval category. We prove also that **TOP** and several subcategories of **POSET** are interval categories. This is an interesting result since **TOP** is not a subcategory of **POSET**.

2 Some Basic Results on Category Theory

Definition 2.1
Let A and B be objects of a category $\mathcal{C}$. A **cartesian product** of A and B is an object $A \times B$ of $\mathcal{C}$ together with two projection morphisms $\pi_0 : A \times B \to A$ and $\pi_1 : A \times B \to B$ satisfying the following universal property: for all object C of $\mathcal{C}$ and all morphisms $f : C \to A$ and $g : C \to B$ there exist a unique morphism $h : C \to A \times B$ making the following diagram

$$\begin{tikzcd}
 & C \arrow[dl, "f"'] \arrow[d, "h"] \arrow[dr, "g"] & \\
A & A \times B \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] & B
\end{tikzcd}$$

commutative.

Definition 2.2
We say a category $\mathcal{C}$ is a **category with cartesian product** if for every ordered pair A, B of objects of $\mathcal{C}$ the cartesian product $A \times B$ exist in $\mathcal{C}$.

Lemma 2.1
Let $\mathcal{C}$ be a category with product. Let $f : A \longrightarrow B$ and $g : C \longrightarrow D$ be morphisms of C. Then there is a unique morphism

$$f \times g : A \times C \longrightarrow B \times D$$

which makes commutative the following diagram

$$\begin{tikzcd}
A \arrow[d, "f"'] & A \times C \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] \arrow[d, "f \times g"] & C \arrow[d, "g"] \\
B & B \times D \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] & D.
\end{tikzcd}$$

Proof: Follows from the universal property of the cartesian product. ■

Remark: The lemma 2.1 guarantees that we have a covariant functor

$$Prod : \mathcal{C} \longrightarrow \mathcal{C}$$

defined by $Prod(A) = A \times A$ for each object A of $\mathcal{C}$ and if $f : A \longrightarrow B$ is a morphism then

$$Prod(f) = f \times f : A \times A \longrightarrow B \times B$$

Definition 2.3
Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ and $G : \mathcal{C} \longrightarrow \mathcal{D}$ be functors. A collection of morphisms $\sigma = \{\sigma_A : F(A) \longrightarrow G(A) \ / \ A \textit{ is an object of } \mathcal{C}\}$, $\sigma_A : F(A) \longrightarrow G(A)$ of $\mathcal{D}$, indexed by objects A of $\mathcal{C}$ is called a **natural transformation** from F to G if the diagram

$$\begin{tikzcd}
F(A) \arrow[r, "\sigma_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"] \\
F(B) \arrow[r, "\sigma_B"] & G(B)
\end{tikzcd}$$

commutes for all morphism $f : A \longrightarrow B$ in $\mathcal{C}$.

Moore information on category theory could be found in [3, 2].

3 The Interval Mathematic

We will denote by $\mathbb{I}(\mathbb{R})$ the set $\{[r, s] \ / \ r, s \in \mathbb{R} \text{ e } r \leq_{\mathbb{R}} s\}$ called the real intervals set. Each interval could be seen as an ordered pair or as a set ($[r, s] = \{x \in \mathbb{R} \ / \ r \leq_{\mathbb{R}} x \leq_{\mathbb{R}} s\}$). The Moore theory, guarantees that all interesting constructions on intervals can be obtained through their extremes.

There are several orders which can be defined on $\mathbb{I}(\mathbb{R})$:

1. **The Kulish-Miranker order [8, 9]:**
$$\begin{aligned} [a, b] \sqsubseteq [c, d] &\Leftrightarrow \forall x \in [a, b] \ \exists y \in [c, d], \ x \leq_{\mathbb{R}} y \text{ and } \forall y \in [c, d] \ \exists x \in [a, b], \ x \leq_{\mathbb{R}} y \\ &\Leftrightarrow a \leq_{\mathbb{R}} c \text{ and } b \leq_{\mathbb{R}} d \end{aligned}$$

2. **The Moore order [10, 11]:**
$$\begin{aligned} [a, b] <_M [c, d] &\Leftrightarrow \forall x \in [a, b] \ \forall y \in [c, d], \ x <_{\mathbb{R}} y \\ &\Leftrightarrow b <_{\mathbb{R}} c \end{aligned}$$
Thus, $[a, b] \leq_M [c, d] \Leftrightarrow [a, b] <_M [c, d]$ or $[a, b] = [c, d]$.

3. **The set-theoretical order:**
$$\begin{aligned} [a, b] \leq_S [c, d] &\Leftrightarrow [a, b] \subseteq [c, d] \\ &\Leftrightarrow c \leq_{\mathbb{R}} a \text{ and } b \leq_{\mathbb{R}} d \end{aligned}$$

4. **The information order [12, 1]:**

$$\begin{aligned}[a,b] \leq_I [c,d] &\Leftrightarrow [c,d] \subseteq [a,b] \\ &\Leftrightarrow a \leq_{\mathbb{R}} c \leq_{\mathbb{R}} d \leq_{\mathbb{R}} b\end{aligned}$$

In this work we will take in account the order of Kulish-Miranker since it is compatible with the order of the cartesian product.

4 The Interval Constructor on POSET

Notice that both the real intervals and the order on the set of real intervals depend upon the usual real order. Thus, we can generalize this constructions by considering, instead of the real set with its usual order, any partially order set, it means that we can think of intervals as a constructor on the category **POSET**.

Definition 4.1
Let $\mathbf{D} = (D, \leq)$ be a poset. The poset $I(\mathbf{D}) = (I(D), \sqsubseteq)$, where

- $I(D) = \{[a,b] \ / \ a, b \in D \text{ and } a \leq b\}$
- $[a,b] \sqsubseteq [c,d] \Leftrightarrow a \leq c \text{ and } b \leq d$

is called the **poset of intervals of D** [4].

There are two natural functions from $I(D)$ to D, which are the left and right projections $l : I(D) \longrightarrow D$ and $r : I(D) \longrightarrow D$ respectively, defined by

$$l([a,b]) = a \text{ and } r([a,b]) = b$$

Clearly the functions l and r are monotonics, therefore they are morphisms from the poset $I(\mathbf{D})$ to the poset $\mathbf{D}$.

Lemma 4.1
Let $\mathbf{D}_1 = (D_1, \leq_1)$ and $\mathbf{D}_2 = (D_2, \leq_2)$ be posets. Then $\mathbf{D}_1 \times \mathbf{D}_2 = (D_1 \times D_2, \leq)$ is a poset and $I(\mathbf{D}_1 \times \mathbf{D}_2)$ is isomorphic to $I(\mathbf{D}_1) \times I(\mathbf{D}_2)$, where $(a,b) \leq (c,d) \Leftrightarrow a \leq_1 c$ and $b \leq_2 d$.

Proof: Since **POSET** is a cartesian closed category, $\mathbf{D}_1 \times \mathbf{D}_2$ is a poset. Let

$$f : I(D_1 \times D_2) \longrightarrow I(D_1) \times I(D_2)$$

be defined by $f([(a,b),(c,d)]) = ([a,c],[b,d])$. Clearly f is monotonic and bijective, whose inverse is also monotonic. Hence f is an isomorphism. ■

Lemma 4.2
Let $\mathbf{D} = (D, \leq)$ be a poset. There is a unique monomorphism $m(\mathbf{D}) : I(D) \longrightarrow D \times D$ which makes the following diagram

$$\begin{tikzcd}
 & I(D) \arrow[dl, "l"'] \arrow[d, "m(\mathbf{D})"] \arrow[dr, "r"] & \\
D & D \times D \arrow[l, "\pi_0"'] \arrow[r, "\pi_1"] & D
\end{tikzcd}$$

commutative.

Proof: Follows from the universal property of the cartesian product. ■

Proposition 4.1
Let $\mathbf{D}_1 = (D_1, \leq_1)$ and $\mathbf{D}_2 = (D_2, \leq_2)$ be posets. Let $f : D_1 \longrightarrow D_2$ be a monotonic function. The function $I(f) : I(D_1) \longrightarrow I(D_2)$, defined by $I(f)([a,b]) = [f(a), f(b)]$, is the unique monotonic function which makes the following diagram

$$\begin{tikzcd}
D_1 \arrow[r, "f"] & D_2 \\
I(D_1) \arrow[u, "l"] \arrow[r, "I(f)"] \arrow[d, "r"'] & I(D_2) \arrow[u, "l"'] \arrow[d, "r"] \\
D_1 \arrow[r, "f"] & D_2
\end{tikzcd}$$

commutative.

Proof: Clearly $I(f)$ is monotonic and makes the above diagram commutative.
If $G : I(D_1) \longrightarrow I(D_2)$ is another monotonic function such that $l \circ G = f \circ l$ and $r \circ G = f \circ r$ then,

$$\begin{aligned}
G([a,b]) = [c,d] \quad & \Leftrightarrow l(G([a,b])) = l([c,d]) \text{ and } r(G([a,b])) = r([c,d]) \\
& \quad\ \text{(since } m(\mathbf{D}_2) : I(D_2) \longrightarrow D_2 \times D_2 \text{ is injective)} \\
& \Leftrightarrow f(l([a,b])) = l([c,d]) \text{ and } f(r([a,b])) = r([c,d]) \\
& \quad\ \text{(by commutativity)} \\
& \Leftrightarrow f(a) = c \text{ and } f(b) = d
\end{aligned}$$

Thus $G([a,b]) = [f(a), f(b)] = I(f)([a,b])$. Therefore $I(f)$ is unique. ■

Remark: The above proposition guarantees that we have a covariant functor $I : \mathbf{POSET} \longrightarrow \mathbf{POSET}$.

Lemma 4.3
The collection

$$m = \{m(\mathbf{D}) : I(D) \longrightarrow Prod(D) \ / \ \mathbf{D} \text{ is a poset } \}$$

of morphisms is a natural transformation from $I : \mathbf{POSET} \longrightarrow \mathbf{POSET}$ to $Prod : \mathbf{POSET} \longrightarrow \mathbf{POSET}$.

Proof: Let $f : D_1 \longrightarrow D_2$ be a monotonic function of posets. We must prove that the following diagram

$$\begin{tikzcd}
I(D_1) \arrow[r, "m(\mathbf{D}_1)"] \arrow[d, "I(f)"'] & D_1 \times D_1 \arrow[d, "f \times f"] \\
I(D_2) \arrow[r, "m(\mathbf{D}_2)"] & D_2 \times D_2
\end{tikzcd}$$

commutes.

In fact, if $[a, b] \in I(D_1)$ then

$$\begin{aligned}
f \times f(m(\mathbf{D}_1)([a, b])) &= f \times f([a, b]) \\
&= (f(a), f(b)) \\
&= m(\mathbf{D}_2)([f(a), f(b)]) \\
&= m(\mathbf{D}_2)(I(f)([a, b]))
\end{aligned}$$

Thus, $(f \times f) \circ m(\mathbf{D}_1) = m(\mathbf{D}_2) \circ I(f)$. ■

5 Interval Categories

Definition 5.1
An **interval category** is a triple $(\mathcal{C}, \mathbb{I}, m)$ such that

1. $\mathcal{C}$ is a category with product
2. $\mathbb{I} : \mathcal{C} \longrightarrow \mathcal{C}$ is a covariant functor such that $\mathbb{I}(A \times B)$ is isomorphic to $\mathbb{I}(A) \times \mathbb{I}(B)$ for all pair of objects A and B of $\mathcal{C}$
3. m is an injective natural transformation from $\mathbb{I} : \mathcal{C} \longrightarrow \mathcal{C}$ to $Prod : \mathcal{C} \longrightarrow \mathcal{C}$
4. There exists a covariant functor $F : \mathcal{C} \longrightarrow \mathbf{POSET}$ such that for each $A, B \in Obj_{\mathcal{C}}$ and for each $f : A \longrightarrow B$ morphism we have that
   (a) $F(\mathbb{I}(A)) = I(F(A))$,
   (b) $F(\mathbb{I}(f)) = I(F(f))$,
   (c) $F(A \times A) = F(A) \times F(A)$,

(d) $F(f \times f) = F(f) \times F(f)$ and

(e) $F(m(A)) = m(F(A))$.

Proposition 5.1
$(\mathbf{POSET}, I, m)$ is an interval category.

Proof: Properties 1), 2) and 3) of the above definition follow from what we had discussed before. To prove property 4) it is enough to take $F : \mathbf{POSET} \longrightarrow \mathbf{POSET}$ as the identity functor. ∎

In the next section we will give a non-trivial example of an interval category.

6 TOP as an Interval category

In this section we will show that the category **TOP**, of topological spaces as objects and continuous maps as morphisms, is an interval category. For more information on topological spaces see [5, 13].

6.1 The functor $\mathbb{I}$

Let X be a topological space. We define the set $\mathbb{I}(X)$ by

$$\mathbb{I}(X) = \{[x, y] \ / \ x, y \in X \text{ and } y \in \mathcal{U} \text{ for each } \mathcal{U} \subseteq X \text{ open set such that } x \in \mathcal{U}\}$$

where $[x, y]$ must be understood as an ordered pair.

Definition 6.1
$\mathcal{U} \subseteq \mathbb{I}(X)$ is **open** if there exist $\mathcal{V}, \mathcal{W} \subseteq X$ open sets such that $\mathcal{W} \subseteq \mathcal{V}$ and

$$\mathcal{U} = \{[x, y] \in \mathbb{I}(X) \ / \ x \in \mathcal{V} \text{ and } y \in \mathcal{W}\}.$$

Let $\mathcal{T} = \{\mathcal{U} \subseteq \mathbb{I}(X) \ / \ \mathcal{U} \text{ is open in } \mathbb{I}(X)\}$.

Lemma 6.1
The pair $(\mathbb{I}(X), \mathcal{T})$ is a topological space.

Proof: If $\{\mathcal{U}_\alpha\}_{\alpha \in A}$ is a family of elements of $\mathcal{T}$ then for each $\alpha \in A$, there are open sets $\mathcal{V}_\alpha, \mathcal{W}_\alpha \subseteq X$ such that $\mathcal{W}_\alpha \subseteq \mathcal{V}_\alpha$ and $\mathcal{U}_\alpha = \{[x, y] \in \mathbb{I}(X) \ / \ x \in \mathcal{V}_\alpha \text{ and } y \in \mathcal{W}_\alpha\}$.

Let $\mathcal{U} = \bigcup_{\alpha \in A} \mathcal{U}_\alpha$, $\mathcal{V} = \bigcup_{\alpha \in A} \mathcal{V}_\alpha$ and $\mathcal{W} = \bigcup_{\alpha \in A} \mathcal{W}_\alpha$. Clearly $\mathcal{W} \subseteq \mathcal{V}$ and $\mathcal{U} = \{[x, y] \in \mathbb{I}(X) \ / \ x \in \mathcal{V} \text{ and } y \in \mathcal{W}\}$ are open sets in X. Thus, $\mathcal{T}$ is closed under arbitrary unions.

Analogously $\mathcal{T}$ is closed under finite intersections. ∎

Let X and Y be topological spaces and $f : X \longrightarrow Y$ be a continuous function. Define $\mathbb{I}(f) : \mathbb{I}(X) \longrightarrow \mathbb{I}(Y)$ by

$$\mathbb{I}(f)([x, y]) = [f(x), f(y)].$$

Lemma 6.2
$\mathbb{I}(f)$ is a continuous function.

Proof: We must prove first that $\mathbb{I}(f)$ is a well defined function, i.e. if $[x,y] \in \mathbb{I}(X)$ then $[f(x), f(y)] \in \mathbb{I}(Y)$. In fact, let $\mathcal{U} \subseteq Y$ be an open sets such that $f(x) \in \mathcal{U}$.

Since $\mathcal{U} \subseteq Y$ is open and $f : X \longrightarrow Y$ is continuous we have that $f^{-1}(\mathcal{U}) \subseteq X$ is open.

Since $f(x) \in \mathcal{U}$, $x \in f^{-1}(\mathcal{U})$ and therefore $y \in f^{-1}(\mathcal{U})$ because $[x,y] \in \mathbb{I}(X)$. Therefore $[f(x), f(y)] \in \mathbb{I}(Y)$ as we wanted.

We will prove next that $\mathbb{I}(f) : \mathbb{I}(X) \longrightarrow \mathbb{I}(Y)$ is continuous.

Let $\mathcal{U} \subseteq \mathbb{I}(Y)$ be an open set. Then, there exists $\mathcal{V}, \mathcal{W} \subseteq Y$ open sets such that $\mathcal{W} \subseteq \mathcal{V}$ and $\mathcal{U} = \{[a,b] \in \mathbb{I}(Y) \ / \ a \in \mathcal{V} \text{ and } b \in \mathcal{W}\}$.

Therefore

$$\begin{aligned} [x,y] \in \mathbb{I}(f)^{-1}(\mathcal{U}) &\Leftrightarrow \mathbb{I}(f)([x,y]) \in \mathcal{U} \\ &\Leftrightarrow [f(x), f(y)] \in \mathcal{U} \\ &\Leftrightarrow f(x) \in \mathcal{V} \text{ and } f(y) \in \mathcal{W} \\ &\Leftrightarrow x \in f^{-1}(\mathcal{V}) \text{ and } y \in f^{-1}(\mathcal{W}). \end{aligned}$$

Thus,

$$\mathbb{I}(f)^{-1}(\mathcal{U}) = \{[x,y] \in \mathbb{I}(X) \ / \ x \in f^{-1}(\mathcal{V}) \text{ and } y \in f^{-1}(\mathcal{W})\}.$$

Since f is continuous, $f^{-1}(\mathcal{W}) \subseteq f^{-1}(\mathcal{V})$ are open sets in X. Hence, $\mathbb{I}(f)^{-1}(\mathcal{U})$ is open in $\mathbb{I}(X)$. Therefore $\mathbb{I}(f)$ is a continuous map. ■

What we had proved so far is that $\mathbb{I} : \mathbf{TOP} \longrightarrow \mathbf{TOP}$ is a covariant functor.

Lemma 6.3
Let X and Y be topological spaces. Then $\mathbb{I}(X \times Y)$ is isomorphic to $\mathbb{I}(X) \times \mathbb{I}(Y)$.

Proof: Let $f : \mathbb{I}(X \times Y) \longrightarrow \mathbb{I}(X) \times \mathbb{I}(Y)$ be defined by

$$f([(a,b),(c,d)]) = ([a,c],[b,d]).$$

It is easy to show that f is a well defined continuous isomorphism. ■

6.2 The Natural Transformation

Let X be a topological space. Define $m(X) : \mathbb{I}(X) \longrightarrow X \times X$ by

$$m(X)([x,y]) = (x,y).$$

Lemma 6.4
$m(X) : \mathbb{I}(X) \longrightarrow X \times X$ is continuous.

Proof: Let $\mathcal{U} \subseteq X \times X$ be a basic open set. Then there exist $\mathcal{V}, \mathcal{W} \subseteq X$ open sets such that $\mathcal{U} = \mathcal{V} \times \mathcal{W}$.

Notice that if $[x,y] \in \mathbb{I}(X)$ then

$$\begin{aligned}[x,y] \in m(X)^{-1}(\mathcal{U}) &\Leftrightarrow (x,y) \in \mathcal{U} \\ &\Leftrightarrow x \in \mathcal{V} \text{ and } y \in \mathcal{W}.\end{aligned}$$

Thus,

$$m(X)^{-1}(\mathcal{U}) = \{[x,y] \in \mathbb{I}(X) \;/\; x \in \mathcal{V} \text{ and } y \in \mathcal{W}\}.$$

Notice that if $[x,y] \in m(X)^{-1}(\mathcal{U})$ then $y \in \mathcal{V} \cap \mathcal{W}$. Thus,

$$m(X)^{-1}(\mathcal{U}) = \{[x,y] \in \mathbb{I}(X) \;/\; x \in \mathcal{V} \text{ and } y \in \mathcal{V} \cap \mathcal{W}\}.$$

Therefore $m(X)^{-1}(\mathcal{U}) \subseteq \mathbb{I}(X)$ is open. Hence $m(X)$ is continuous. ■

Lemma 6.5
Let X and Y be topological spaces. Let $f : X \longrightarrow Y$ be a continuous map. Then the following diagram

$$\begin{tikzcd}
\mathbb{I}(X) \arrow[r, "m(X)"] \arrow[d, "\mathbb{I}(f)"'] & X \times X \arrow[d, "f \times f"] \\
\mathbb{I}(Y) \arrow[r, "m(Y)"] & Y \times Y
\end{tikzcd}$$

commutes.

Proof: Let $[x,y] \in \mathbb{I}(X)$. Then

$$\begin{aligned} f \times f(m(X)([x,y])) &= (f(x), f(y)) \\ &= m(Y)([f(x), f(y)]) \\ &= m(Y)(\mathbb{I}(f)([x,y])).\end{aligned}$$

Hence, $(f \times f) \circ m(X) = m(Y) \circ \mathbb{I}(f)$.
Therefore the above diagram commutes. ■

The above lemmas proved that m is a natural transformation from the functor $\mathbb{I}$ to the functor $Prod$.

6.3 The Functor F

Let X be a topological space. Define the poset $F(X) = (X, \leq_X)$ where

$$x \leq_X y \Leftrightarrow y \in \mathcal{U} \text{ for each } \mathcal{U} \subseteq X \text{ open set such that } x \in \mathcal{U}.$$

It is well known that this order is a partial order on X [13].

Lemma 6.6
If $f : X \longrightarrow Y$ is a continuous map then f is monotonic with respect to the above partial order.

Proof: Let $x, y \in X$ such that $x \leq_X y$. Let $\mathcal{U} \subseteq Y$ be an open set such that $f(x) \in \mathcal{U}$. Hence $f^{-1}(\mathcal{U}) \subseteq X$ is open and $x \in f^{-1}(\mathcal{U})$.

Since $x \leq_X y$, $y \in f^{-1}(\mathcal{U})$. Hence, $f(y) \in \mathcal{U}$.

Therefore, $f(x) \leq_Y f(y)$. ■

Let $f : X \longrightarrow Y$ be a continuous map. Define $F(f) = f$. The above lemma guarantees that $F(f)$ is a morphism from the poset $F(X)$ to the poset $F(Y)$. Thus $F : \mathbf{TOP} \longrightarrow \mathbf{POSET}$ is a covariant functor.

Proposition 6.1
$(\mathbf{TOP}, \mathbb{I}, m)$ is an interval category.

Proof: Properties 1), 2) and 3) of definition 5.1 follows from the above discussions. Property 4) follows straightforward from the definition of F, m and $\mathbb{I}$. We will only show that $F(\mathbb{I}(X)) = I(F(X))$ for all topological space X.

Notice that

$$
\begin{aligned}
[x, y] \leq_{\mathbb{I}(X)} [u, v] \quad & \Leftrightarrow [u, v] \in \mathcal{U} \text{ for all } \mathcal{U} \subseteq \mathbb{I}(X) \text{ open set such that} \\
& \quad\ [x, y] \in \mathcal{U} \\
& \Leftrightarrow u \in \mathcal{V} \text{ and } v \in \mathcal{W} \text{ for all } \mathcal{V}, \mathcal{W} \subseteq X \text{ open sets} \\
& \quad\ \text{such that } \mathcal{W} \subseteq \mathcal{V},\ x \in \mathcal{V} \text{ and } y \in \mathcal{W} \\
& \Leftrightarrow x \leq_X u \text{ and } y \leq_Y v \\
& \Leftrightarrow [x, y] \sqsubseteq [u, v] \text{ in } I(F(X)).
\end{aligned}
$$

■

7 Some Others Interval Categories

In this section we study some domains categories which are interval categories.

Lemma 7.1
Let $(\mathcal{C}, \mathbb{I}, m)$ be an interval category. If $\mathcal{D}$ is a subcategory of $\mathcal{C}$ with cartesian product, which is closed under the functor $\mathbb{I}$, then $(\mathcal{D}, \mathbb{I}', m')$ is an interval category, where $\mathbb{I}'$ and m' are the restrictions of $\mathbb{I}$ and m to $\mathcal{D}$.

Proof: Straightforward. ■

Corollary 7.1
The following subcategories of **POSET** are interval categories:

1. **DCPO** with dcpos as objects and continuous functions as morphisms.

2. **CCDCPO** with consistently complete dcpos as objects and continuous functions as morphisms.

3. **ADCPO** with algebraic dcpos as objects and continuous functions as morphisms.

4. **SDom** with Scott domains as objects and continuous functions as morphisms.

5. **CDCPO** with continuous dcpos as objects and continuous functions as morphisms.

6. **CDom** with continuous domain as objects and continuous functions as morphisms.

Proof: It is enough to show that all those categories are closed under the functor I. This follows straightforward from the definition of those categories (more information about those categories could be found in [6, 7]) and from the definition of the functor I in **POSET**. ■

8 Conclusions

We defined interval categories in order to generalize usual results of the Moore interval theory to other kind of categories. We showed that some important categories are interval categories, such as the category **TOP** of topological spaces with continuous functions, and several domain categories. This is an interesting result since **TOP** is not a subcategory of **POSET**. As a biproduct we define an interval constructor I on the **POSET** category which generalizes the interval constructor on the real set. This result is important in order to have a formal treatment of parametric interval data type. Therefore, we are given a theoretical foundations to develop programming languages which have the parametric interval data type as primitive type.

Further works:

1. To consider interval categories based on the other orders.

2. To define interval categories in an intrinsic way

3. To prove that the interval functor is well behaved under other domain constructors.

4. To extend the interval arithmetic to the **POSET** category.

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