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Comparing continuity of interval functions based on Moore and Scott topologies

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Abstract

In this paper we prove some facts about the relation between Moore and Scott topologies on intervals. We also show how those topologies can be recovered from a given quasi-metric and how topological real line is perfectly embedded into both topologies, meaning that Scott and Moore topologies are extensions of Euclidean line.

Keywords: *Interval functions, continuity, Moore-topology, Scott topology*

1 Introduction

Continuity is a fundamental notion in real analysis it can be recovered by using more abstract notions; namely distance and topology.

On the other hand, in interval mathematics, introduced by T. Sunaga in [20] and by R.E.Moore in [11] a real number is represented by intervals of float-points instead of a unique float-point. Interval arithmetics guarantees that the expected real number, i.e. the ideal output, of a numerical algorithm is in the output interval of the respective interval algorithm. The mathematical theory of intervals generalizes the usual real mathematics, therefore a good notion of continuity for interval functions is required in order to obtain an interval analysis which extends real analysis. Moore in [12] proposed a notion of continuity based on a metric distance, the resulting topology is Hausdorff. This topology, roughly speaking, is the sub-topology of Euclidean plane $\mathbb{R} \times \mathbb{R}$. Therefore intervals can be seen as a subspace of the real plane. In fact most of publication on interval analysis use this topology.

On the other hand, in [15], Dana Scott considers intervals with reverse inclusion order as a continuous domain (Consistently complete continuous dcpo). Since, each domain has an associated topology, known as Scott topology, and an equivalent order based continuity, then intervals receives another notion of continuity, named here as Scott continuity. In [2], B. Acióly used this notion of continuity to provide a computational foundation for interval mathematics. This viewpoint provides an information theory for intervals; namely they are interpreted as partial information of real numbers. In [1], B. Acióly and B. Bedregal provide a quasi-metric approach for Scott continuity of intervals and give a first comparison for Moore and Scott continuities. In this paper, we will extend this comparison in such a way that we can establish the relation between both class of continuous functions. We also include several, not original, but interesting results; e.g. some relations between these topologies and usual real topology.

2 Some topological notions

The idea of continuity can be generalized and expressed on the language of set theory, i.e. in terms of subsets, unions, intersections, complements, etc. The resulting theory is called **topology**. To make this paper self-contained, we will give a fast introduction to some topological concepts. For more detailed information about topology the reader can see [6, 17].

2.1 Topologies.

Given a non-emptyset A, a collection $\Omega \subseteq \mathcal{P}(A)$ [1] is a **topology** on A, if $\emptyset, A \in \Omega$ and for every $S_1, S_2 \in \Omega$ and $\mathcal{T} \subseteq \Omega$, $S_1 \cap S_2 \in \Omega$ and $\bigcup \mathcal{T} \in \Omega$. Every element of a topology is called an **open set** and the pair (A, Ω) is called a **topological space**.The pair $(\mathbb{R}, \Omega)$, where for every $O \in \Omega$ there exist a family $\mathcal{F}$ of open intervals (a, b) such that $O = \bigcup \mathcal{F}$ is a topological space called **Euclidean topology**.

Let (A, Ω) be a topological space, and let Y be a subset of X. The **relative topology** for Y is the collection $\{G \cap Y : G \in \Omega\}$.

It is possible to define different topologies on the same set A, depending on the way we want to classify its points. Sometimes open sets can be very complicated but can be described by using a selection of fairly simple open sets called **basic open sets**. When this happen, the set of those simple open sets

[1] $\mathcal{P}(A)$ is the powerset of A.

is called a **base** for that topology. Rigorously, speaking, a family of open sets $\mathcal{B} \subseteq \Omega$ is a **base for** Ω whenever for every non-empty open subset $O \in \Omega$, there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that $O = \bigcup_{X \in \mathcal{B}'} X$.

2.2 Continuity and homeomorphism.

Transformations which preserves topological properties are called continuous functions. So, given two topological spaces (A, Ω_1) and (B, Ω_2), a **continuous function** $f : A \to B$ is such that for all $O \in \Omega_2$, $f^{-1}(O) \in \Omega_1$.

The idea of equivalent objects up to their nature, known in Algebra as isomorphic structures (i.e. both objects share the same algebraic properties), is also present in topology through the idea of homeomorphic spaces: Given two topological spaces (A, Ω_1) and (B, Ω_2), a function $f : A \to B$ is an **homeomorphism**, if it is a continuous bijective function such that its inverse f^{-1} is also continuous. In that case, (A, Ω_1) and (B, Ω_2) are said to be **homeomorphic spaces** and, as in algebra, they have the same topological properties.

2.3 Connectedness and intervals

Some of the following results will require the notion of **connectedness** on metric spaces and the effect of continuous functions on intervals. Here we define it for topological spaces:

A subset $S \subseteq A$ of a topological space [2] is a **disconnected set** if there are two disjoint open sets U and V such that: (1) $U \cup V \supseteq S$; and (2) $U \cap S$ and $V \cap S$ are both non-empty. S is a **connected set** if it is not disconnected. For connectedness on real numbers, the idea of intervals is very important. A set $I \subseteq \mathbb{R}$ is an interval [3] if whenever $x, y \in I$ and $x < y$ then every $z \in \mathbb{R}$ which satisfies $x \leq z \leq y$ also belongs to I. The following lemmas highlight some well-known connections between intervals and connectedness.

Lemma 2.1 *A subset of $A \subseteq \mathbb{R}$ is connected iff A is an interval.*

Lemma 2.2 *Continuous functions preserves connectedness; i.e. if (A, Ω_1) and (B, Ω_2) are topological spaces and $f : A \to B$ is a continuous function, then for all connected subset $C \subseteq A$, $f(C) \in \Omega_2$ is also connected.*

Assuming Euclidean topology on $\mathbb{R}$, from lemmas 2.1 and 2.2 we easily derive corollary 2.1 and lemma 2.3:

Corollary 2.1 *If $f : \mathbb{R} \to \mathbb{R}$ is continuous and $I \subseteq \mathbb{R}$ is an interval, then $f(I)$ is also an interval.*

Observe that this corollary does not guarantee that any kind of interval is mapped in an interval of the same kind — e.g. it does not mean that $f([a, b]) = [c, d]$. However, there are some conditions such that closed intervals are mapped to closed intervals by continuous functions; namely:

Lemma 2.3 *If $f : \mathbb{R} \to \mathbb{R}$ is continuous, $I \subseteq \mathbb{R}$ is a closed interval, and there exist $x_0, x_1 \in I$ such that for each $z \in I$, $f(x_0) \leq f(z) \leq f(x_1)$, then $f(I)$ is also a closed interval.*

[2] By abuse of language, since a topological space is a pair (A, Ω).

[3] An interval can be a set of the form $(a, b), [a, b], (a, b], [a, +\infty)$, etc.

Proof: By corollary 2.1 $f(I)$ is an interval. Since $x_0, x_1 \in I$, then, trivially, $f(x_0), f(x_1) \in f(I)$ and by hypothesis for all $z \in I$, $f(x_0) \leq f(z) \leq f(x_1)$. So trivially,$f(I) = [f(x_0), f(x_1)]$ and therefore $f(I)$ is a closed interval. ■

From now on we use the word "interval" to designate, just, closed intervals — i.e. sets of the form $[a, b] \subseteq \mathbb{R}$.

3 Topologies for Moore intervals and their relations with real numbers

In this section we give some new results about the application of Scott topology and quasi-metric spaces on the set of Moore intervals. The idea is to establish a connection between Moore Topology, Scott Topology and Euclidean Topology. The subsections 3.1 and 3.2 show that the topological space of real numbers (the Euclidean space) is perfectly embedded in Moore and Scott topological spaces, respectively, and subsection 4 establishes a connection between Scott and Moore topologies for intervals into a quasi-metric environment.

3.1 Metrics and Moore Topology

The concept of distance in a set A is formalized by the notion of **metric**, which is a function $d : A \times A \to \mathbb{R}$, such that for all $a, b, c \in A$, $d(a, b) = 0 \Leftrightarrow a = b$, $d(a, b) = d(b, a)$, and $d(a, c) \leq d(a, b) + d(b, c)$. The pair (A, d) is called **metric space**. On the set of real numbers and on the set of intervals the notion of distance between two real numbers and two intervals, is given, respectively, by the functions $dr(r, s) = \mid r - s \mid$ and $di([a, b], [c, d]) = max(dr(a, c), dr(b, d))$. Those metrics are called **Euclidean metric** and **Moore metric** [13], respectively. The pairs $(\mathbb{R}, dr)$ and $(\mathbb{I}(\mathbb{R}), di)$ are called, respectively, **metric spaces** of real numbers and intervals.

Given a metric space (A, d) it is possible to define a topological space (A, Ω_d) induced by d, where the basic open sets are called **open balls**: $B(a, \epsilon) = \{s \in A : d(a, s) < \epsilon\}$. The resulting topological space induced by dr is exactly the Euclidean topology, whose basic open sets are open intervals $(a - \epsilon, a + \epsilon)$. An important fact, is that the set of degenerated intervals $Tot(\mathbb{I}(\mathbb{R})) = \{[a, a] \in \mathbb{I}(\mathbb{R}) : a \in \mathbb{R}\}$ endowed with di, induces the topological space $(Tot(\mathbb{I}(\mathbb{R})), \Omega_{di})$ which is *homeomorphic to Euclidean space* $(\mathbb{R}, \Omega_{dr})$, which means, in some sense, that the Euclidean space is represented (embedded) into $(\mathbb{I}(\mathbb{R}), \Omega_{di})$.

A function $f : A \to B$, where (A, d) and (B, d') are metric spaces, is called **continuous at** $a \in A$ if, for every $\epsilon > 0$, there is $\delta > 0$, such that for every $x \in A$, if $d(x, a) < \delta$, then $d'(f(x), f(a)) < \epsilon$. f is a **continuous function**, if it is continuous in every $a \in A$.

This notion of continuity in metric spaces generalizes for topological continuity mentioned above and, in some sense, for ord-continuity bellow. Ord-continuity generalizes the idea of convergent sequences to directed sets and the preservation of limits to the preservation of supremum. In what follows, we show this concept of continuity, its relation with topological continuity and its application to Moore intervals.

3.2 Scott-continuity and Ord-continuity

It is well known, that the notion of convergence in Moore topology, does not match with that of inclusion monotonicity. In what follows we present another topology for intervals, where those notions match. It is Scott topology, and, in the case of intervals, the notion of convergence is based on nesting intervals. The main results of this section are: (1) lemma 3.2 and proposition 3.2, which show the matching between inclusion monotonicity and topological convergence; and (2) theorem 3.1 that shows, like in Moore topology, an embedding of Euclidean space into this topology. In what follows we develop a little of domain theory to be able to define Scott topology and to prove some results.

The theory of partially ordered sets has been used along the time for semantics of programming languages [7, 16, 19] and in some other fields of computing. A **partially ordered set** is a pair $(A, \leq)$ where $\leq$ is a reflexive, transitive, and antisymmetric binary relation on A called **partial order on** A.

The idea of information applied to semantics of programming languages is suitably modelled by partial orders, where converging sequences of information and limit of those sequences can be defined, respectively, in terms of **directed sets** and **supremum of directed sets**. A directed set, Δ, is a non-empty set such that for every pair $x, y \in \Delta$, there is $z \in \Delta$ such that $x \leq z$ and $y \leq z$, and a supremum of a directed set, $\bigsqcup \Delta$, is its **least upperbound** [4]. A partially ordered set where every directed set has supremum is called **directed complete partial order**, or just **dcpo**. Those partial orders extend the well known property of real numbers, that every convergent sequence has limit. The idea of continuous function on real numbers — where the limits of convergent sequences are preserved — is generalized to dcpo's in terms of the preservation of supremum. Therefore, a function $f : A \to B$, where A and B are dcpo's is an **ord-continuous function** if for every directed set Δ, $f(\bigsqcup \Delta) = \bigsqcup f(\Delta)$. It is easy to verify that ord-continuous functions are also monotonic[5].

Definition 3.1 *Let be the set of closed intervals* $\mathbb{I}(\mathbb{R}) = \{[a, b] : a, b \in \mathbb{R}$ *and* $a \leq b\}$ *and the following partial order: For all* $[a, b], [c, d] \in \mathbb{I}(\mathbb{R})$,

$$[a, b] \sqsubseteq [c, d] \textit{ iff } a \leq c \leq d \leq b. \tag{1}$$

Proposition 3.1 $\langle \mathbb{I}(\mathbb{R}), \sqsubseteq \rangle$ *is a dcpo.*

Proof: Trivially $\sqsubseteq$ is a partial order. In fact it coincide with $\supseteq$ when see intervals as sets of real numbers.

It is well known that a poset is a dcpo if and only if every chain has a supremum [5, 8, 10].

Let $\Delta \subseteq \mathbb{I}(\mathbb{R})$ be a chain. Then trivially, if $[a, b] \in \Delta$ then b is an upper bound of $\pi_1(\Delta) = \{x \in \mathbb{R} : [x, y] \in \Delta\}$ and a is lower bound of $\pi_2(\Delta) = \{y : [x, y] \in \Delta\}$. So, by order completeness property of real numbers, $\pi_1(\Delta)$ has a supreme in $(\mathbb{R}, \leq)$ and $\pi_2(\Delta)$ has an infimum. Clearly, $\bigsqcup \Delta = [\bigsqcup \pi_1(\Delta), \bigsqcap \pi_2(\Delta)]$. ■

Moore intervals in $(\mathbb{I}(\mathbb{R}), \sqsubseteq)$ are, therefore, considered partial information of real numbers. The intuition is that any interval $[a, b]$ is a partial information of a degenerated interval $[r, r]$, whenever $r \in [a, b]$, and $[r, r]$ is a **totally defined object**, since it only informs about itself.

[4] i.e. (1) for every $d \in \Delta$, $d \leq \bigsqcup \Delta$; and (2) for every u such that for every $d \in \Delta$, $d \leq u$, $\bigsqcup \Delta \leq u$.

[5] A function $f : A \to B$, where A and B are partially ordered sets is monotonic if $x \leq y$ implies $f(x) \leq f(y)$.

Definition 3.2 *Given any dcpo $(A, \leq)$, it is possible to define a topology called* ***Scott-topology****,* $\Omega_S(A)$ *whose open sets has the following properties: For all $O \in \Omega_S(A)$,*

1. *if $x \in O$ and $x \leq y$, then $y \in O$*[6]*; and*
2. *if Δ is a directed set and $\bigsqcup \Delta \in O$, then $\Delta \cap O \neq \emptyset$.*

Lemma 3.1 *If D is a dcpo, then the set $U_x = \{z \in D : z \not\leq x\}$*[7] *is a Scott open set.*

Proof: If $y \in U_x$ and $y \leq w$, then $y \not\leq x$ and consequently $w \not\leq x$ and $w \in U_x$. Suppose that $\bigsqcup \Delta \in U_x$, where Δ is a directed set, i.e. $\bigsqcup \Delta \not\leq x$. If $U_x \cap \Delta = \emptyset$, then for each $y \in \Delta$, $y \leq x$ (x is an upperbound of Δ) and $\bigsqcup \Delta \leq x$, what is a contradiction. So $U_x \cap \Delta \neq \emptyset$. ∎

Lemma 3.2 *Let $f : A \to B$ be a function, where A and B are dcpo's, if f is continuous with respect to Scott topologies (we abbreviate by saying that f is* ***Scott-continuous****), then it is monotonic.*

Proof: Suppose that f is not monotone, then for some x and y, $x \leq y$ and $f(x) \not\leq f(y)$, by definition $f(x) \in U_{f(y)}$ and therefore $x \in f^{-1}(U_{f(y)})$. Since $f^{-1}(U_{f(y)})$ is a Scott-open set, then $y \in f^{-1}(U_{f(y)})$ and $f(y) \in U_{f(y)}$, what is a contradiction since $f(y) \leq f(y)$. ∎

Proposition 3.2 *A function $f : A \to B$, where A and B are dcpo's, is ord-continuous iff it is Scott-continuous.*

Proof: ($\Rightarrow$) Suppose that f is ord-continuous, i.e. $f(\bigsqcup \Delta) = \bigsqcup f(\Delta)$, where Δ is directed. We must show that $f^{-1}(O) \in \Omega_S(A)$, whenever $O \in \Omega_S(B)$; that is it satisfies the conditions of a Scott open set: (1) If $x \in f^{-1}(O)$ and $x \leq y$ then $f(x) \leq f(y)$, because f is monotonic. Since $f(x) \in O$, then because O is a Scott open set $f(y) \in O$ and therefore $y \in f^{-1}(O)$. (2) If $\bigsqcup \Delta \in f^{-1}(O)$, then $f(\bigsqcup \Delta) = \bigsqcup f(\Delta) \in O$. Therefore, since O is a Scott open set $f(\Delta) \cap O \neq \emptyset$, and so $\Delta \cap f^{-1}(O) \neq \emptyset$.
($\Leftarrow$) Suppose that f is Scott-continuous, we must prove that $f(\bigsqcup \Delta) = \bigsqcup f(\Delta)$, where Δ is directed. By lemma 3.2 $f(\Delta)$ is a directed set for all directed set Δ and $\bigsqcup f(\Delta) \leq f(\bigsqcup \Delta)$. Now suppose that $f(\bigsqcup \Delta) \not\leq \bigsqcup f(\Delta)$, then $f(\bigsqcup \Delta) \in U_{\bigsqcup f(\Delta)}$, which is a Scott-open set. Since f is Scott-continuous then $\bigsqcup \Delta \in f^{-1}(U_{\bigsqcup f(\Delta)})$ which is also an open set. Because Δ is directed and $f^{-1}(U_{\bigsqcup f(\Delta)})$ is a Scott open set, then $\Delta \cap f^{-1}(U_{\bigsqcup f(\Delta)}) \neq \emptyset$; i.e. there exist k such that $k \in \Delta$, $k \in f^{-1}(U_{\bigsqcup f(\Delta)})$, $f(k) \in f(\Delta)$, and $f(k) \in U_{\bigsqcup f(\Delta)}$. Therefore, $f(k) \leq \bigsqcup f(\Delta)$ and $f(k) \not\leq \bigsqcup f(\Delta)$ — contradiction. ∎

This proposition means that the notion of topological continuity can be expressed in terms of order, more precisely in terms of inclusion monotonicity which preserves the supremum and vice versa. We close this subsection with theorem 3.1 which shows that Euclidean topology can be extended to Scott-topology.

Lemma 3.3 *The set $\Uparrow\mathbb{I}(\mathbb{Q}) = \{\Uparrow[a, b] : [a, b] \in \mathbb{I}(\mathbb{Q})\}$ where $\Uparrow[a, b] = \{[c, d] \in \mathbb{I}(\mathbb{R}) : a < c \leq d < b\}$ is a base for $\Omega_S(\mathbb{I}(\mathbb{R}))$.*

[6] i.e. O is upperclosed.
[7] i.e. the collection of all points that either lie above x or is incomparable with x.

Proof: First we will prove that $\Uparrow[a,b]$ is a Scott open set.

If $[c,d] \in \Uparrow[a,b]$ and $[c,d] \sqsubseteq [e,f]$ then $a < c$, $d < b$, $c \leq e$ and $f \leq d$. So, $a < e$ and $f < b$. Therefore, $[e,f] \in \Uparrow[a,b]$. If Δ is a directed set and $\bigsqcup \Delta = [u,v] \in \Uparrow[a,b]$ then $a < u$ and $v < b$. So, $[u,v] \in \Uparrow[a,b]$. Thus, if $[u,v] \in \Delta$ then $\Delta \cap \Uparrow[a,b] \neq \emptyset$. If $[u,v] \notin \Delta$ and for each $[c,d] \in \Delta$ such that $c \leq a$, then $[a,v]$ is an upper bound of Δ and therefore $[u,v] \sqsubseteq [a,v]$. So, $u \leq a$. But, by hypothesis $a < u$, which is a contradiction. Therefore, if $[u,v] \notin \Delta$ then there exists $[c,d] \in \Delta$ such that $a < c$. Analogously, we can prove that if $[u,v] \notin \Delta$ then there exists $[c',d'] \in \Delta$ such that $d' < b$. Since Δ is directed, $\exists [e,f] \in \Delta$ such that $[c,d] \sqsubseteq [e,f]$ and $[c',d'] \sqsubseteq [e,f]$. So, because $a < c < e$ and $f < d' < b$, $[e,f] \in \Uparrow[a,b]$ and therefore $\Delta \cap \Uparrow[a,b] \neq \emptyset$.

Let $O \in \Omega_S(\mathbb{I}(\mathbb{R}))$. Then trivially, $O = \bigcup_{[a,b]\in O} \Uparrow[a,b]$. So, $\Uparrow\mathbb{I}(\mathbb{R})$ is a base for $\Omega_S(\mathbb{I}(\mathbb{R}))$. ■

Lemma 3.4 *Let be $Tot(\mathbb{I}(\mathbb{R}))$, **the set of degenerated intervals**, and the family $\Omega_S(Tot(\mathbb{I}(\mathbb{R})))$, **the relative topology of** $\Omega_S(\mathbb{I}(\mathbb{R}))$ **restricted to** $Tot(\mathbb{I}(\mathbb{R}))$. Then, the set $B = \{([x,x],[y,y]) : x, y \in \mathbb{R}$ and $x < y\}$ where $([x,x],[y,y]) = \{[z,z] : x < z < y\}$ is a base for $\Omega_S(Tot(\mathbb{I}(\mathbb{R})))$.*

Proof: Let $O \in \Omega_S(Tot(\mathbb{I}(\mathbb{R})))$ then by definition, there exist $O' \in \Omega_S(\mathbb{I}(\mathbb{R}))$ such that $O' \cap Tot(\mathbb{I}(\mathbb{R})) = O$. By lemma 3.3 and definition of base, there exists $S \subseteq \mathbb{I}(\mathbb{Q})$ such that $O' = \bigcup_{[x,y]\in S} \Uparrow[x,y]$. Since, $\Uparrow[x,y] \cap Tot(\mathbb{I}(\mathbb{R})) = ([x,x],[y,y])$, then $O' \cap Tot(\mathbb{I}(\mathbb{R})) = \bigcup_{[x,y]\in S}([x,x],[y,y])$ and therefore $O = \bigcup_{[x,y]\in S}([x,x],[y,y])$. ■

Theorem 3.1 *Let $Tot(\mathbb{I}(\mathbb{R}))$ be the set of degenerated intervals, $\Omega_S(Tot(\mathbb{I}(\mathbb{R})))$ the relative topology of $\Omega_S(\mathbb{I}(\mathbb{R}))$ restricted to $Tot(\mathbb{I}(\mathbb{R}))$, and $\mathcal{U}$ the usual Euclidean topology on $\mathbb{R}$. Then, there is an homeomorphism $f : Tot(\mathbb{I}(\mathbb{R})) \to \mathbb{R}$.*

Proof: Let $f : Tot(\mathbb{I}(\mathbb{R})) \to \mathbb{R}$ defined by $f([x,x]) = x$. If $O \in \Omega_S(Tot(\mathbb{I}(\mathbb{R})))$ then, by lemma 3.4, $O = \bigcup_{[x,y]\in A}([x,x],[y,y])$ for some $A \subseteq \mathbb{I}(\mathbb{Q})$. So, $f(O) = \{f([z,z]) : [z,z] \in ([x,x],[y,y]) \subseteq O\} = \bigcup_{[x,y]\in A}(x,y) \in \mathcal{U}$. Trivially, $f^{-1} : \mathbb{R} \to Tot(\mathbb{I}(\mathbb{R}))$ is defined by $f^{-1}(x) = [x,x]$. Clearly, $f^{-1}((x,y)) = \{f^{-1}(z) : z \in (x,y)\} = \{[z,z] : z \in (x,y)\} = ([x,x],[y,y])$. If $O \in \mathcal{U}$ then $O = \bigcup_{(x,y)\in A}(x,y)$ for some $A \subseteq \{(p,q) : p,q \in \mathbb{Q}$ and $p < q\}$. So, $f^{-1}(O) = f^{-1}(\bigcup_{(x,y)\in A}(x,y)) = \bigcup_{(x,y)\in A} f^{-1}((x,y)) = \bigcup_{(x,y)\in A}([x,x],[y,y])) \in \Omega_S(Tot(\mathbb{I}(\mathbb{R})))$. So, f and f^{-1} are continuous. Since $f \circ f^{-1} = id$ and $f^{-1} \circ f = id$ then f is bijective. Therefore, f^{-1} is continuous. Thus, f is an homeomorphism. ■

3.3 An ϵ-δ approach

In the last section we proved that both Scott and Moore topologies can be used to represent[8] the topological Euclidean line. However, it is not clear how can we use the ϵ-δ approach in Scott-topology. In what follows we show that both Moore and Scott topologies can be defined in terms of a specific quasi-metric [18]. Quasi-metric spaces generalize metric spaces, and therefore, enable us to use the ϵ-δ approach, both in Scott and Moore spaces. This subsection also prepares the next one, where we characterize both the class of Moore and Scott continuous functions.

[8]The idea of representation of spaces can be found in [21] and is applied to the notion of interval correctness in [14].

A **quasi-metric** generalizes the notion of metric, in the sense that it is a function $d : A \times A \to \mathbb{R}$, such that (a) $d(a,a) = 0$, (b)$d(a,c) \leq d(a,b) + d(b,c)$ and (c) $d(a,b) = d(b,a) = 0 \Rightarrow a = b$[9]. A **quasi-metric space** is a pair (A,q), where A is a set and q a quasi-metric over A. For every quasi-metric q, it is always possible to define another quasi-metric called **conjugated quasi-metric** defined by $\overline{q}(a,b) = q(b,a)$ and a metric q^*, such that $q^*(a,b) = max\{q(a,b), \overline{q}(a,b)\}$. Since q generalizes the notion of distance, it is also possible to define open balls $B(a,\epsilon) = \{s \in A : q(a,s) < \epsilon\}$, $\overline{B}(a,\epsilon) = \{s \in A : \overline{q}(a,s) < \epsilon\}$ and $B^*(a,\epsilon) = \{s \in A : q^*(a,s) < \epsilon\}$. These three kinds of balls define two topological spaces induced from quasi-metrics $\mathcal{T}(q)$ and $\mathcal{T}(\overline{q})$, and a topological space, $\mathcal{T}(q^*)$, induced by the metric q^*[10].

Lemma 3.5 *A subset O of a (quasi-)metric space A is an open set if and only if for all $x \in O$, there exist $\epsilon > 0$, such that $B(x,\epsilon) \subseteq O$.*

Proof: Straightforward. ■

Lemma 3.6 *The function $qi : \mathbb{I}(\mathbb{R}) \times \mathbb{I}(\mathbb{R}) \to \mathbb{R}$, defined by*

$$qi([a,b],[c,d]) = max\{c-a, b-d, 0\} \tag{2}$$

is a quasi-metric for $\mathbb{I}(\mathbb{R})$.

Proof: See [1]. ■

In what follows we show that this quasi-metric, when extended to qi^* coincides exactly with the Moore-metric.

Proposition 3.3 *For all $[a,b],[c,d] \in \mathbb{I}(\mathbb{R})$, $qi^*([a,b],[c,d]) = di([a,b],[c,d])$.*

Proof:

$$\begin{aligned} qi^*([a,b],[c,d]) &= max\{qi([a,b],[c,d]), \overline{qi}([a,b],[c,d])\} \\ &= max\{max\{c-a, b-d, 0\}, max\{a-c, d-b, 0\}\} \\ &= max\{c-a, b-d, a-c, d-b, 0\} \\ &= max\{\mid c-a \mid, \mid b-d \mid\} \\ &= di([a,b],[c,d]). \end{aligned}$$

■

Definition 3.3 *Let (A,q) and (B,q') be quasi-metric spaces. A function $f : A \to B$[11] is called **quasi-continuous at** $a \in A$ if, for every $\epsilon > 0$, there is $\delta > 0$, such that for every $x \in A$, if $q(x,a) < \delta$, then $q'(f(x),f(a)) < \epsilon$. f is a **quasi-continuous function** if it is quasi-continuous for every $a \in A$. A quasi-continuous function $f : (A,q) \to (B,q')$ is called **bi-continuous** if it is also quasi-continuous with respect to $\overline{q}$ and $\overline{q'}$.*

[9] Observe that $d(a,b) = d(b,a) = 0$ and $a \neq b$ does not hold, but $d(a,b) = d(b,a) \neq 0$ and $a = b$, and $d(a,b) = d(b,a) \neq 0$ and $a \neq b$ can hold.

[10] Note that when q is a metric, $q, \overline{q}$, and q^* coincide and the topological spaces are the same.

[11] Sometimes we will use the notation $f : (A,\Sigma) \to (B,\Sigma')$, where Σ and Σ' are structures on the sets A and B — e.g. metrics, quasi-metrics, topologies, operations, etc. — to avoid ambiguity and redundancies in the language. When the context is enough we omit the structure part for question of clarity.

Proposition 3.4 *$f : (A, q) \to (B, q')$ is quasi-continuous at a if and only if for all open set $O' \in \mathcal{T}(q')$ such that $f(a) \in O'$, there exist $O \in \mathcal{T}(q)$, where $a \in O$ and $f(O) \subseteq O'$.*

Proof: ($\Rightarrow$) Suppose that f is quasi-continuous at $a \in A$ and $f(a) \in O' \in \mathcal{T}(q')$. By lemma 3.5 there exist $\epsilon > 0$ such that $B(f(a), \epsilon) \subseteq O'$. Since f is quasi-continuous then there exist $\delta > 0$ such that for every $x \in A$, if $q(x, a) < \delta$, then $q'(f(x), f(a)) < \epsilon$, i.e. if $x \in B(a, \delta)$, then $f(x) \in B(f(a), \epsilon)$. Therefore, $f(B(a, \delta)) \subseteq B(f(a), \epsilon)$.

($\Leftarrow$) Suppose that for all open set $O' \in \mathcal{T}(q')$ such that $f(a) \in O'$, there exist $O \in \mathcal{T}(q)$, where $a \in O$ and $f(O) \subseteq O'$. Let $\epsilon > 0$ and $B(f(a), \epsilon)$, therefore by hypothesis there exist $O \in \mathcal{T}(q)$ such that $a \in O$ and $f(O) \subseteq B(f(a), \epsilon)$. By lemma 3.5 there exist $\delta > 0$ such that $B(a, \delta) \subseteq O$. For each $x \in A$, if $q(a, x) < \delta$, then $x \in B(a, \delta)$. Since $B(a, \delta) \subseteq O$ then $f(B(a, \delta)) \subseteq f(O) \subseteq B(f(a), \epsilon)$. Since, $f(x) \in f(B(a, \delta))$ then $f(x) \in B(f(a), \epsilon)$, i.e. $q'(f(a), f(x)) < \epsilon$. So, f is quasi-continuous. ∎

Since every metric is quasi-metric, this proposition and the following also hold for metrics and conjugated quasi-metrics.

Proposition 3.5 *Given two quasi-metric spaces (A, q) and (B, q'). $f : (A, q) \to (B, q')$ is quasi-continuous if and only if $f : (A, \mathcal{T}(q)) \to (B, \mathcal{T}(q'))$ is continuous.*

Proof: ($\Rightarrow$) Suppose that $f : (A, q) \to (B, q')$ is quasi-continuous. Given $O' \in \mathcal{T}(q')$, we will show that $f^{-1}(O') \in \mathcal{T}(q)$. For each $a \in f^{-1}(O')$, we have that $f(a) \in O'$. So, by lemma 3.5, there exist $\epsilon > 0$ such that $B(f(a), \epsilon) \subseteq O'$. Since f is quasi-continuous at a, then there exist $\delta > 0$ such that $f(B(a, \delta)) \subseteq B(f(a), \epsilon) \subseteq O'$. Therefore, $\bigcup_{a \in f^{-1}(O')} B(a, \delta) \subseteq f^{-1}(O')$. Since $a \in B(a, \delta)$, then for all $a \in f^{-1}(O')$, $a \in \bigcup_{a \in f^{-1}(O')} B(a, \delta)$, so $\bigcup_{a \in f^{-1}(O')} B(a, \delta) = f^{-1}(O')$.

($\Leftarrow$) Suppose that $f^{-1}(O') \in \mathcal{T}(q)$, for each $O' \in \mathcal{T}(q')$. Let $a \in A$. We will show that $f : (A, q) \to (B, q')$ is quasi-continuous at a. Given $\epsilon > 0$, the open ball $B(f(a), \epsilon) \in \mathcal{T}(q')$. Thus $f^{-1}(B(f(a), \epsilon)) \in \mathcal{T}(q)$. Since $f(a) \in B(f(a), \epsilon)$, then $f^{-1}(f(a)) = a \in f^{-1}(B(f(a), \epsilon))$. So, there exist $\delta > 0$ such that $B(a, \delta) \subseteq f^{-1}(B(f(a), \epsilon))$. That is $f(B(a, \delta)) \subseteq B(f(a), \epsilon)$. Therefore, according to proposition 3.4, f is quasi-continuous. ∎

Lemma 3.7 *Let (A, q) and (B, q') be quasi-metric spaces. If $f : (A, q) \to (B, q')$ is bi-continuous then $f : (A, q^*) \to (B, q'^*)$ is continuous.*

Proof: If $f : A \to B$ is quasi-continuous function w.r.t. q and q', then for each $a \in A$ and for each $\epsilon > 0$ there exists δ_a such that for each $b \in A$ if $q(b, a) < \delta_a$ then $q'(f(b), f(a)) < \epsilon$. Analogously, If $f : A \to B$ is quasi-continuous function w.r.t. $\overline{q}$ and $\overline{q'}$, then for each $a \in A$ and for each $\epsilon > 0$ there exists $\overline{\delta}_a$ such that for each $b \in A$ if $\overline{q}(b, a) < \overline{\delta}_a$ then $\overline{q'}(f(b), f(a)) < \epsilon$.

Thus, for each $a \in A$ and $\epsilon > 0$ define $\delta = min(\delta_a, \overline{\delta}_a)$. Let $b \in A$ such that $q^*(b, a) < \delta$, then $q^*(b, a) < \delta_a$ and $q^*(b, a) < \overline{\delta}_a$. So, by definition of q^*, $q(b, a) < \delta_a$ and $\overline{q}(b, a) < \overline{\delta}_a$. Hence, $q'(f(b), f(a)) < \epsilon$ and $\overline{q'}(f(b), f(a)) < \epsilon$ and therefore $q'^*(f(b), f(a)) < \epsilon$. So, f is continuous w.r.t. q^* and q'^* . ∎

Theorem 3.2 *The topology $\mathcal{T}(q_i)$ is homeomorphic to $\Omega_S(\mathbb{I}(\mathbb{R}))$.*

Proof: [c.f. [1]] It is sufficient to show that the base of the standard topology induced by qi (the open ϵ-balls $B(x,\epsilon)$) is a basis (or equivalently contain a basis) of $\Omega_S(\mathbb{I}(\mathbb{R}))$. Since we know that $\{\Uparrow[p,q] : [p,q] \in \mathbb{I}(\mathbb{Q})\}$ is a basis of $\Omega_S(\mathbb{I}(\mathbb{R}))$, we will show that $\Uparrow[p,q]$ is an open ϵ-ball. Let $B([p+\frac{q-p}{3}, q-\frac{q-p}{3}], \frac{q-p}{3})$.

For each $[r,s] \in \mathbb{I}(\mathbb{R})$, if $[p,q] \ll [r,s]$ then or $p < r \leq q - \frac{q-p}{3}$ or $p + \frac{q-p}{3} \leq s < q$. In both case, we have that $max\{r-p+\frac{q-p}{3}, q-\frac{q-p}{3}-s, 0\} < \frac{q-p}{3}$. So, $qi([p+\frac{q-p}{3}, q-\frac{q-p}{3}], [r,s]) < \frac{q-p}{3}$. Therefore if $[r,s] \in \Uparrow[p,q]$ then $[r,s] \in B([p+\frac{q-p}{3}, q-\frac{q-p}{3}], \frac{q-p}{3})$. On the other hand, for each $[r,s] \in \mathbb{I}(\mathbb{R})$, if $[p,q] \not\ll [r,s]$ then or $r \leq p$ or $q \leq s$ and therefore $max\{r-p+\frac{q-p}{3}, q-\frac{q-p}{3}-s, 0\} \geq \frac{q-p}{3}$. So if $[r,s] \notin \Uparrow[p,q]$ then $[r,s] \notin B([p+\frac{q-p}{3}, q-\frac{q-p}{3}], \frac{q-p}{3})$. Hence, $\Uparrow[p,q] = B([p+\frac{q-p}{3}, q-\frac{q-p}{3}], \frac{q-p}{3})$.

■

Corollary 3.1 *$F : (\mathbb{I}(\mathbb{R}), qi) \to (\mathbb{I}(\mathbb{R}), qi)$ is quasi-continuous if and only if $F : (\mathbb{I}(\mathbb{R}), \Omega_S(\mathbb{I}(\mathbb{R}))) \to (\mathbb{I}(\mathbb{R}), \Omega_S(\mathbb{I}(\mathbb{R})))$ is Scott-continuous.*

Proof: Straightforward from proposition 3.5 and theorem 3.2. ■

Corollary 3.2 *$F : (\mathbb{I}(\mathbb{R}), qi) \to (\mathbb{I}(\mathbb{R}), qi)$ is bi-continuous then:*

1. *$F : (\mathbb{I}(\mathbb{R}), di) \to (\mathbb{I}(\mathbb{R}), di)$ is Moore-continuous; and*
2. *$F : (\mathbb{I}(\mathbb{R}), \Omega_S(\mathbb{I}(\mathbb{R}))) \to (\mathbb{I}(\mathbb{R}), \Omega_S(\mathbb{I}(\mathbb{R})))$ is Scott-continuous.*

Proof:

1. Straightforward from proposition 3.3 and lemma 3.7;
2. From definition of bi-continuity and corollary 3.1.

■

Therefore the class of bi-continuous interval functions are included in the intersection of Moore with Scott continuous functions. In the next section we show how the classes of Moore and Scott continuous functions are related.

4 Moore-continuity vs. Scott-continuity

In the sequel we compare Moore and Scott-continuities. The result is a classification for interval functions in terms of those topologies. The main result is that *the family of Moore-continuous functions are not included into Scott-continuous family and vice-versa*, but they are intersecting families of interval functions which contain the class of **bi-continuous functions**.

4.1 Moore-continuity does not implies Scott-continuity

According to [13]:

" Not all interval valued functions, however, are inclusion monotonic. For example, consider the interval function F defined by

$$F(X) = m(X) + \frac{1}{2}(X - m(X))\text{''}, \tag{3}$$

where $m([a,b]) = \frac{a+b}{2}$ — the midpoint of $[a,b]$ — and $w([a,b]) = b - a$ — the width of $[a,b]$.

Since it is not inclusion monotonic, then it is not Scott-continuous (see Proposition 3.2). In [1] is stated that this function is Moore-continuous, however it is not proved. In what follows we prove that to justify the title of this subsection.

Lemma 4.1 *Let F be the function (3). Then, for each $X, Y \in \mathbb{I}(\mathbb{R})$,*

$$di(F(X), F(Y)) \leq 2di(X, Y).$$

Proof: First notice that if $X = [\underline{x}, \overline{x}]$ then

$$F(X) = \left[\frac{\underline{x} + m(X)}{2}, \frac{\overline{x} + m(X)}{2}\right] = \left[\underline{x} + \frac{w(X)}{4}, \overline{x} - \frac{w(X)}{4}\right]$$

So, by definition of di,

$$\begin{aligned} di(F(X), F(Y)) &= \max\{\mid \underline{y} + \tfrac{w(Y)}{4} - (\underline{x} + \tfrac{w(X)}{4}) \mid, \mid \overline{y} - \tfrac{w(Y)}{4} - (\overline{x} - \tfrac{w(X)}{4}) \mid\} \\ &= \max\{\mid \underline{y} - \underline{x} + \tfrac{w(Y) - w(X)}{4} \mid, \mid \overline{y} - \overline{x} + \tfrac{w(X) - w(Y)}{4} \mid\} \end{aligned}$$

If $X \subseteq Y$, then $\underline{x} - \underline{y} \geq 0$ and

$$\begin{aligned} w(Y) - w(X) &= (\overline{y} - \underline{y}) - (\overline{x} - \underline{x}) \\ &= (\overline{y} - \overline{x}) - (\underline{y} - \underline{x}) \\ &= (\overline{y} - \overline{x}) + (\underline{x} - \underline{y}) \\ &= \mid \overline{y} - \overline{x} \mid + \mid \underline{y} - \underline{x} \mid \\ &\geq 0. \end{aligned}$$

If $\overline{y} - \overline{x} \leq \underline{x} - \underline{y}$ (the case for $\underline{x} - \underline{y} \leq \overline{y} - \overline{x}$ is analogous) then $di(X, Y) = \underline{x} - \underline{y}$. Thus,

$$\begin{aligned} \frac{w(Y) - w(X)}{4} &= \frac{(\overline{y} - \overline{x}) + (\underline{x} - \underline{y})}{4} \\ &= \frac{\overline{y} - \overline{x}}{4} + \frac{\underline{x} - \underline{y}}{4} \\ &\leq \frac{\overline{x} - \overline{y}}{4} + \frac{\underline{x} - \underline{y}}{4} \\ &= \frac{\underline{x} - \underline{y}}{2} \end{aligned}$$

So, $\dfrac{w(Y) - w(X)}{4} \leq di(X, Y)$. Therefore $\overline{y} - \overline{x} + \frac{w(X) - w(Y)}{4} \leq 2di(X, Y)$ and $\underline{y} - \underline{x} + \dfrac{w(X) - w(Y)}{4} \leq 2di(X, Y)$. Hence, $di(F(X), F(Y)) \leq 2di(X, Y)$.

Suppose that $X \leq_{KM} Y$, where $\leq_{KM}$ is the Kulisch-Miranker order [9][12]. (1) Case $\mid \underline{y} - \underline{x} \mid \leq \mid \overline{y} - \overline{x} \mid$. Then $0 \leq w(Y) - w(X)$ and

$$\begin{aligned} w(Y) - w(X) &= \overline{y} - \underline{y} - (\overline{x} - \underline{x}) \\ &= \overline{y} - \overline{x} - (\underline{y} - \underline{x}) \\ &= \mid \overline{y} - \overline{x} \mid - \mid \underline{y} - \underline{x} \mid \\ &\leq \mid \overline{y} - \overline{x} \mid \end{aligned}$$

[12] $X \leq_{KM} Y$ iff $\underline{x} \leq \underline{y}$ and $\overline{x} \leq \overline{y}$.

and therefore $\frac{w(Y)-w(X)}{4} \leq \mid \overline{y} - \overline{x} \mid$. By hypothesis $\mid \underline{y} - \underline{x} \mid \leq \mid \overline{y} - \overline{x} \mid$, hence $\mid \underline{y} - \underline{x} + \frac{w(Y)-w(X)}{4} \mid \leq 2 \cdot \mid \overline{y} - \overline{x} \mid = 2 \cdot di(X,Y)$. Since $w(X) \leq w(Y)$ then $\frac{w(X)-w(Y)}{4} \leq 0$. So $\mid \overline{y} - \overline{x} + \frac{w(X)-w(Y)}{4} \mid \leq 2 \cdot \mid \overline{y} - \overline{x} \mid = 2 \cdot di(X,Y)$. Therefore, $di(F(X), F(Y)) \leq 2 \cdot di(X,Y)$. (2) Case $\mid \overline{y} - \overline{x} \mid \leq \mid \underline{y} - \underline{x} \mid$ then $w(Y) \leq w(X)$. So, $w(Y) - w(X) \leq 0$ and therefore $\mid \underline{y} - \underline{x} + \frac{w(Y)-w(X)}{4} \mid = \underline{y} - \underline{x} + \frac{w(Y)-w(X)}{4} \leq \mid \underline{y} - \underline{x} \mid \leq 2 \cdot di(X,Y)$. On the other hand, $0 \leq w(X) - w(Y)$ and

$$\begin{aligned} w(X) - w(Y) &= \overline{x} - \underline{x} - (\overline{y} - \underline{y}) \\ &= \underline{y} - \underline{x} - (\overline{y} - \overline{x}) \\ &= \mid \underline{y} - \underline{x} \mid - \mid \overline{y} - \overline{x} \mid \leq \mid \underline{y} - \underline{x} \mid \\ &= di(X,Y). \end{aligned}$$

Since, by hypothesis $0 \leq \overline{y} - \overline{x} \leq \underline{y} - \underline{x}$ then $\mid \overline{y} - \overline{x} + \frac{w(X)-w(Y)}{4} \mid = \overline{y} - \overline{x} + \frac{w(X)-w(Y)}{4} \leq 2 \cdot \mid \underline{y} - \underline{x} \mid = 2 \cdot di(X,Y)$. Therefore, $di(F(X), F(Y)) \leq 2di(X,Y)$.

The other cases ($Y \subseteq X$ and $Y \leq_{KM} X$) are proved considering the symmetry of di and the previous cases. ■

Proposition 4.1 *The function F defined in equation 3 is Moore-continuous.*

Proof: Let $X \in \mathbb{I}(\mathbb{R})$ and $\epsilon > 0$. Define $\delta = \frac{\epsilon}{2}$. If $Y \in \mathbb{I}(\mathbb{R})$ is such that $di(X,Y) < \delta$ then, by lemma 4.1, $di(F(X), F(Y)) \leq 2 \cdot di(X,Y) \leq 2 \cdot \delta = 2 \cdot \frac{\epsilon}{2} = \epsilon$, i.e. $di(F(X), F(Y)) \leq \epsilon$. Therefore, F is Moore-continuous. ■

Theorem 4.1 *There exist an interval function which is Moore-continuous but not Scott-continuous.*

Proof: By proposition 4.1, the function F in equation 3 is Moore-continuous. But this function is not inclusion monotonic and therefore not Scott continuous (see Proposition 3.2). ■

4.2 Scott-continuity does not implies Moore-continuity

Proposition 4.2 *There exists an interval function which is Scott continuous but is not Moore continuous.*

Proof: Let $F : \mathbb{I}(\mathbb{R}) \longrightarrow \mathbb{I}(\mathbb{R})$ defined by

$$F(X) = \begin{cases} [-1,1] & \text{, if } 0 \in X \\ [0,0] & \text{, otherwise} \end{cases}$$

F is Scott continuous: Trivially is monotonic. Let Δ be a directed set. If $0 \in \bigsqcup \Delta$ then $F(\bigsqcup \Delta) = [-1,1]$ and for each $X \in \Delta$, $0 \in X$, and therefore $F(X) = [-1,1]$. So, $\bigsqcup F(\Delta) = [-1,1]$. If $0 \notin \bigsqcup \Delta$ then $F(\bigsqcup \Delta) = [0,0]$ and for some $X \in \Delta$, $0 \notin X$, and therefore $F(X) = [0,0]$. So, $\bigsqcup F(\Delta) = [0,0]$.

F is not Moore-continuous: Clearly,

$$\begin{aligned} F^{-1}(B([-1,1], 0.5)) &= \{X \in \mathbb{I}(\mathbb{R}) : di(F(X), [-1,1]) < 0.5\} \\ &= \{X \in \mathbb{I}(\mathbb{R}) : 0 \in X\}. \end{aligned} \quad (4)$$

Thus, $[0,0] \in F^{-1}(B([-1,1], 0.5))$, but if $F^{-1}(B([-1,1], 0.5))$ was a Moore open ball, there must exists a basic open ball $B([a,b], \epsilon)$ contained in $F^{-1}(B([-1,1], 0.5))$ and containing $[0,0]$. By equation

(4), $0 \in [a,b]$. Since $di([a,b],[0,0]) < \epsilon$, then $0 \leq -a < \epsilon$ and $0 \leq b < \epsilon$. Thus, $0 < \frac{\epsilon+a}{2} < \epsilon$. Therefore $\mid a - \frac{\epsilon+a}{2} \mid = \frac{\epsilon}{2} + \frac{-a}{2} < \epsilon$ and $\mid b - \frac{\epsilon+a}{2} \mid \leq \mid b - \frac{\epsilon}{2} \mid + \frac{-a}{2} < \epsilon$. So, $di([a,b],[\frac{\epsilon+a}{2},\frac{\epsilon+a}{2}]) < \epsilon$ and therefore, $[\frac{\epsilon+a}{2},\frac{\epsilon+a}{2}] \in B([a,b],\epsilon)$. Still, because $0 \notin [\frac{\epsilon+a}{2},\frac{\epsilon+a}{2}]$, $B([a,b],\epsilon) \not\subseteq F^{-1}(B([-1,1],0.5))$ which is a contradiction. ■

5 Final remarks

It is clear that Moore and Scott topologies are not the unique possible topologies defined on Moore intervals. In fact, Bedregal and Santiago [4] introduced an uncountable family of pseudo-metrics for $\mathbb{I}(\mathbb{R})$ — this paper also contains the conjugate of the quasi-metric exposed here. We can also consider the notion of bicontinuity (which, was proved in corollary 3.2, is both Moore and Scott continuous), or the continuity notions proposed in [3], etc. Now, what is the better notion of continuity for intervals? it depends on what viewpoint of intervals we want to explore. If we like consider real intervals as just an order pair of real numbers then Moore topology is more suitable, on the other hand if we would like consider an interval as a representation or as information of a real number then the Scott's viewpoint is better.

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