Interval t-norms as interval representations of t-norms

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***Abstract*— In this paper a new interval generalization for t-norms is proposed. The main property of our generalization is that each interval t-norm can be seen as an interval representation of a t-norm and, therefore, can be used to deal with correctness of interval fuzzy algorithms. Moreover, how to obtain in a canonical way the best interval representation (interval t-norm) for a t-norm will be shown. Also it will be proved that the main classes of t-norms, such as continuous, Archimedean, without zero divisors, and idempotent, are preserved by these constructions.**

I. Introduction

The fuzzy sets theory introduced by Lofti Zadeh in [40] has as its main characteristic to consider a degree of belief, i.e. a real value in $[0,1]$, to indicate how much a specialist may believe that the element belongs to the set. In this way, fuzzy logic, their subjacent logic, becomes an important tool to deal with the uncertainty of knowledge and to represent the uncertainty of human reasoning.

There are two main directions in fuzzy logic [27]:

1) Fuzzy logic in the broad sense, which has as main goal the development of systems based on fuzzy reasoning, such as fuzzy control systems, and
2) Fuzzy logic in the narrow sense, which sees fuzzy logic as a symbolic logic, and therefore questions such as formal theories are considered.

Considerable progress has been made in strictly mathematical (formal and symbolic) aspects of fuzzy logic as logic with a comparative notion of truth [17].

Menger in [28] introduced the triangular norms (t-norms) notion in order to model the distance in probabilistic metric spaces. Schweizer and Sklar in [33] gave an axiomatic for t-norms. In [3], Alsina, Trillas, and Valverde use t-norms and their dual notion (t-conorm) to model conjunction and disjunction connectives in fuzzy logics, generalizing several previous fuzzy interpretations for the conjunction, provided, among others, by Lofti Zadeh in [40], Bellman and Zadeh in [6] and Yager in [38] (which define a general class of interpretations). From a t-norm it also is possible to obtain canonical fuzzy interpretation for implication and negation connectives [7]. Thus, each t-norm determines a different set of true formulas (1-tautologies) and false formulas (0-contradictions) and therefore different fuzzy logics. In this way, t-norms have been hugely responsible for the progress of fuzzy logic in the narrow sense.

On the other hand, interval mathematics was introduced by T. Sunaga in [36] and by R.E.Moore in [29] with the goal of providing a mathematical foundation for interval computations. In interval computations, a real number is represented by intervals with float points as end points instead of simply by float points in such a way that the real output is in the interval output whenever a real input is in an input interval. This property is know as *correctness* [16] and is guaranteed by the principle of maximal exactness (roundoff ”outward”, i.e. rounded down and rounded up) and optimal scalar product [4]. Thus, interval computation gives an automatic and rigorous control of digital error of numerical computations and therefore is adequate to deal with the imprecision in the input values or caused by the roundoff errors which occur during computation [29], [30], [4].

Another important property of interval constructions pointed by Hickey [16] is *optimality*, where an interval operation is optimal w.r.t. a real operation if the interval result is the narrowest possible containing all possible results of the real operation. The correctness and optimality properties were formalized, for the context of interval functions, by Santiago et al. in [32] through the concepts of *interval representation* and *canonical interval representation*, respectively.

A great synergy exists between interval mathematics and fuzzy logic. Several works have explored this synergy in different ways, for example [37], [13], [10], [31], [39], [24], [23]. In [26], Lodwick points out four relationships between fuzzy set theory and interval analysis. The fourth one considers a union of both to provide an analysis of uncertainty. In this kind of relationship intervals are used as membership degree of fuzzy sets, with the goal of dealing with the uncertainty associated with digital computers. This approach is also adequate for addressing the imprecision of a specialist in providing an exact value to measure membership uncertainty. According to Yam et al. in [39]

"if an expert is uncertain about something, he can as well be uncertain about his degree of belief as well, and it is quite possible that the expert will not be able to describe his degree of uncertainty exactly: e.g., a person can meaningfully distinguish between his degrees of belief 0.6 and 07., but hardly anyone will be able to describe his degree of belief as 0.6 and not 0.601."

Moreover, it is hard to believe that a specialist can assign $\sqrt{0.2}$ as a degree of membership of an object to some fuzzy sets.

This paper proposes a generalization of t-norms for interval values. This generalization, different than the other generalization of t-norms [41], [10], [12], is concerned with the correctness and optimality aspect. In this sense, it will provide a canonical construction to obtain an interval t-norm from a t-norm in such a way that it always guarantees that the interval result of the interval t-norm is the narrowest interval containing the real result of the t-norm. Also analyzed which characteristic of the t-norm, such as continuity and Archimedean, are preserved by the respective interval t-norm. Several other intermediary results for interval t-norms are given.

In this paper we will assume that the reader is familiar with t-norm theory. There exists much literature on this topics; we recommend [19], [20], [21], [22]

II. Interval Representations

Let $\mathbb{IR}$ be the set of intervals with real numbers as end points. That is, $\mathbb{IR} = \{[r, s] \ : \ r, s \in \mathbb{R} \text{ and } r \leq s\}$. So, an interval has a dual nature: a set of real numbers and an ordered pair of real numbers. $\mathbb{IR}$ is associated with two projections: $\pi_1 : \mathbb{IR} \longrightarrow \mathbb{R}$ and $\pi_2 : \mathbb{IR} \longrightarrow \mathbb{R}$ defined by

$$\pi_1([\underline{x}, \overline{x}]) = \underline{x} \text{ and } \pi_2([\underline{x}, \overline{x}]) = \overline{x}$$

As convention, for any interval variable X, $\pi_1(X)$, and $\pi_2(X)$ will be denoted by $\underline{x}$ and $\overline{x}$, respectively.

Interval analysis, proposed by T. Sunaga in [36] and by R.E. Moore [29], provides a mathematical framework for computing real functions with an automatic control of computational errors. In this sense, real numbers are **represented** by float-point intervals instead of simple float points, guaranteeing that the resulting interval contains the value of the real function. This property is known as **correctness criterium** [16]. In [32], correctness was formalized thorough the notion of **interval representation**, where an interval function $F : \mathbb{IR}^m \longrightarrow \mathbb{IR}^n$ represents a real function $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ if for each $X \in \mathbb{IR}^m$, $f(x) \in F(X)$ always $x \in X$ (the interval X represents x). If the function f is not asymptotic then the function $\widehat{f} : \mathbb{IR}^m \longrightarrow \mathbb{IR}^n$ defined by

$$\widehat{f}(X) = [\inf\{f(x) \ : \ x \in X\}, \sup\{f(x) \ : \ x \in X\}]$$

is well defined and it is an interval representation of f [32]. So, by definition $\widehat{f}$ is a sharp function [18]. Clearly, if F is also an interval representation of f, then for each $X \in \mathbb{IR}$, $\widehat{f}(X) \subseteq F(X)$. Thus, $\widehat{f}$ returns a narrower interval than any other interval representation of f and is therefore their **best interval representation**. Thus, $\widehat{f}$, when seen as an algorithm to compute f, has the **optimality property** of interval algorithms mentioned by Hickey et al. in [16]. When f is continuous then $\widehat{f}(X) = \{f(x) \ : \ x \in X\}$ [32].

A. Partial orders on $\mathbb{IR}$

Real numbers have a natural total order. However, there is not any total order which extends the real order in a natural way. Nevertheless, some partial orders on $\mathbb{IR}$ consider different natures of intervals.

1) When an interval is seen as a set of real numbers, the natural partial order is the inclusion, introduced by T. Sunaga in [36]. Formally, for each $X, Y \in \mathbb{IR}$,

$$X \subseteq Y \Leftrightarrow \underline{y} \leq \underline{x} \leq \overline{x} \leq \overline{y}.$$

2) When an interval is seen as ordered pair of real numbers, the natural order is compatible with the order of the cartesian product, introduced by Kulisch and Miranker in [25]. Formally, for each $X, Y \in \mathbb{IR}$,

$$X \leq Y \Leftrightarrow \underline{x} \leq \underline{y} \text{ and } \overline{x} \leq \overline{y}.$$

3) When an interval is seen as an information or as a representation of an unknown real number, the natural order is that introduced by Scott in [34] and widely used by Aciółly in [1] to provide a computational foundation of interval mathematics. Formally, for each $X, Y \in \mathbb{IR}$,

$$X \sqsubseteq Y \Leftrightarrow \underline{x} \leq \underline{y} \leq \overline{y} \leq \overline{x}.$$

In this paper we will consider all these orders.

III. Scott and Moore Continuity

There are several topologies on $\mathbb{IR}$ [32] which extend the real topology. Since continuity of a function is related to a topology, these different topologies determine different notions of continuity. In this paper we will consider Scott continuity and Moore continuity.

The Scott domain is a mathematics theory introduced by Dana S. Scott in the late 1960s [34] with the goal of providing denotational semantics for programs. In the first works continuous lattices were considered for this purpose, but from the mid-1970s were more used the algebraic and continuous classes of consistently complete cpos, or simply Scott domains and continuous domains, respectively. This theory was used by some authors (e.g. [1], [5], [9], [8]) to provide a computational foundation for interval mathematics. For unfamiliar readers on domain theory we suggest [15], [35], [14].

In the following we will provide the basic notions of this theory:

Let $\mathbf{D} = \langle D, \leq \rangle$ be a partially ordered set (poset). An element $x \in D$ is **total** if for each $y \in D$, $x \leq y$ implies that $x = y$. A set $\Delta \subseteq D$ is **directed** if $\forall a, b \in \Delta, \ \exists c \in \Delta$

such that $a \leq c$ and $b \leq c$. A poset $\mathbf{D}$ is **directed complete** (**dcpo** for short) if each directed subset Δ has a least upper bound (or supremum), denoted by $\bigsqcup \Delta$, in $\mathbf{D}$. If a dcpo has a least element then it is called a **pointed dcpo** or simply a **cpo**. A subset $X \subseteq D$ is **consistent** if there exists an upper bound in $\mathbf{D}$, i.e. if there exists $x \in D$ such that for each $a \in X$, $a \leq x$. A dcpo $\mathbf{D}$ is **consistently complete** if each nonempty consistent set $X \subseteq D$ has a supremum in D (i.e. $\bigsqcup X$ exists in $\mathbf{D}$). An element x is **way below** an element y, denoted by $x \ll y$, if for each directed set Δ such that $y \leq \bigsqcup \Delta$ there exists $z \in \Delta$ such that $x \leq z$. Let $\mathord{\downarrow\!\!\downarrow} x$ be the set $\{y \in D : y \ll x\}$. A (d)cpo is continuous if for each $x \in D$, $\mathord{\downarrow\!\!\downarrow} x$ is directed and $\bigsqcup \mathord{\downarrow\!\!\downarrow} x = x$. $\mathbf{D}$ is a **continuous domain** if it is a consistently complete continuous cpo.

Let $\mathbf{D} = \langle D, \leq_D\rangle$ and $\mathbf{E} = \langle E, \leq_E\rangle$ be dcpo's. A function $f : E \longrightarrow D$ is called **Scott continuous** if it is monotonic ($x \leq_E y$ implies $f(x) \leq_D f(y)$) and preserves supremum ($f(\bigsqcup \Delta) = \bigsqcup f(\Delta)$).

Each cpo has associated a topology, denominated Scott topology, whose opens are the sets $O \subseteq D$ such that

1) for each $x \in O$ if $x \leq y$ then $y \in O$ and
2) if $\bigsqcup \Delta \in O$ for some directed set Δ then $\Delta \cap O \neq \emptyset$.

A well known fact is that a function $f : D \longrightarrow E$ is Scott continuous if and only if it is topologically continuous w.r.t. the Scott topologies.

The pair $\Re = \langle \mathbb{IR}, \sqsubseteq\rangle$ is a continuous domain and therefore we have a continuity notion on the interval functions based on this order.

Notice that the subjacent idea of a Scott domain theory is that the total elements are the ideal objects whenever non total elements are partial representations of total elements, in this sense if $X \sqsubseteq Y$ then the representation Y is more precise or provides more information of an ideal object than X. The monotonicity of the function guarantees that whenever more precise is the representation of the ideal input, the representation of the ideal output will be more exact. In the case of $\Re$ domain, the total elements are just the degenerate intervals which can be identified with the real numbers. So, in this theory intervals are representations of real numbers that belong to the interval. Clearly, if F is Scott continuous then F is monotonic with respect to the information order and consequently is inclusion monotonic.

On the other hand, an interval function $F : \mathbb{IR} \longrightarrow \mathbb{IR}$ is Moore continuous if it is continuous w.r.t. the usual Moore metric $d([a, b], [c, d]) = \max\{\mid a - c \mid, \mid b - d \mid\}$.

The main result in [32] is the following:

***Theorem** 3.1:* Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. f is continuous iff $\widehat{f}$ is Scott continuous iff $\widehat{f}$ is Moore continuous.

IV. Pseudo t-norms

In spite of the associativity condition of t-norms being very important, see for example [33], when looking at t-norms as commutative monoids, we can relax the definition of t-norm to abolish this condition. This weak kind of t-norm will be named **pseudo t-norm**. An example of a pseudo t-norm (which is not a t-norm) is

$$T(x, y) = \min\{x, y\}\sqrt{\max\{x, y\}}$$

The order on a pseudo t-norm is similar to that on t-norm.

***Proposition** 4.1:* Let T be a pseudo t-norm. Then $T_W \leq T \leq T_G$ [1].

Proof: Suppose that $x \leq y$. Then, by monotonicity, $T(x, y) \leq T(x, 1)$ and by, 1-identity, $T(x, 1) = x$. But, because $x \leq y$, $T_G(x, y) = x$. So, $T(x, y) \leq T_G(x, y)$. The case when $y \leq x$ is analogous.

Let $x, y \in [0, 1]$. If $x = 1$ then, by 1-identity, $T_W(x, y) = y = T(x, y)$. Analogously, if $y = 1$, $T_W(x, y) = y = T(x, y)$. If $x \neq 1$ and $y \neq 1$ then $T_W(x, y) = 0$ and therefore $T_W(x, y) \leq T(x, y)$. ■

V. Interval t-norms

Let $\mathbb{I} = \{X \in \mathbb{IR} : X \subseteq [0, 1]\}$. A mapping $\mathbb{T} : \mathbb{I}\times\mathbb{I} \longrightarrow \mathbb{I}$ is an **interval triangular norm**, **it-norm** in short, if $\mathbb{T}$ satisfies the follow properties:

1) Symmetry: for each $X, Y \in \mathbb{I}$, $\mathbb{T}(X, Y) = \mathbb{T}(Y, X)$,
2) Associativity: for each $X, Y, Z \in \mathbb{I}$, $\mathbb{T}(X, \mathbb{T}(Y, Z)) = \mathbb{T}(\mathbb{T}(X, Y), Z)$,
3) KM-Monotonicity: for each $X_1, Y_1, X_2, Y_2 \in \mathbb{I}$ if $X_1 \leq X_2$ and $Y_1 \leq Y_2$ then $\mathbb{T}(X_1, Y_1) \leq \mathbb{T}(X_2, Y_2)$,
4) Inclusion Monotonicity: for each $X_1, Y_1, X_2, Y_2 \in \mathbb{I}$ if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ then $\mathbb{T}(X_1, Y_1) \subseteq \mathbb{T}(X_2, Y_2)$ and
5) 1-identity: for each $X \in \mathbb{I}$, $\mathbb{T}(X, [1, 1]) = X$.

Notice that if an interval function is inclusion monotonic then it is also monotonic w.r.t. the information order.

A. Obtaining it-norms from t-norms

The following proposition, considering $T_1 = T_2$, shows how to obtain from any t-norm an it-norm.

***Proposition** 5.1:* Let T_1 and T_2 t-norms. If $T_1 \leq T_2$ then $I[T_1, T_2] : \mathbb{I} \times \mathbb{I} \longrightarrow \mathbb{I}$ defined by

$$I[T_1, T_2](X, Y) = [T_1(\underline{x}, \underline{y}), T_2(\overline{x}, \overline{y})]$$

is an it-norm, denominated **it-norm derived** of T_1 and T_2.

Proof: By definition of interval, $\underline{x} \leq \overline{x}$ and $\underline{y} \leq \overline{y}$. So, by monotonicity of t-norms, $T_1(\underline{x}, \underline{y}) \leq T_1(\overline{x}, \overline{y})$. Since $T_1 \leq T_2$, $T_1(\overline{x}, \overline{y}) \leq T_2(\overline{x}, \overline{y})$. So, $T_1(\underline{x}, \underline{y}) \leq T_2(\overline{x}, \overline{y})$. Therefore, $I[T_1, T_2]$ is well defined.

In the sequel it will be proved that $I[T_1, T_2]$ is an it-norm.

Symmetry: $\forall X, Y \in I[0, 1]$

$$\begin{aligned} I[T_1, T_2](X, Y) &= [T_1(\underline{x}, \underline{y}), T_2(\overline{x}, \overline{y})] \\ &= [T_1(\underline{y}, \underline{x}), T_2(\overline{y}, \overline{x})] \\ &= I[T_1, T_2](Y, X) \end{aligned}$$

Associativity: $\forall X, Y, Z \in I[0, 1]$

[1] T_G denotes the Gödel t-norm, also known as minimum t-norm, and T_W is the weak t-norm or drastic product t-norm.

$$\begin{array}{ll} I[T_1,T_2](X,I[T_1,T_2](Y,Z)) & = \\ I[T_1,T_2](X,[T_1(\underline{y},\underline{z}),T_2(\overline{y},\overline{z})]) & = \\ [T_1(\underline{x},T_1(\underline{y},\underline{z})),T_2(\overline{x},T_2(\overline{y},\overline{z}))] & = \\ [T_1(T_1(\underline{x},\underline{y}),\underline{z}),T_2(T_2(\overline{x},\overline{y}),\overline{z})] & = \\ I[T_1,T_2]([T_1(\underline{x},\underline{y}),T_2(\overline{x},\overline{y})],Z) & = \\ I[T_1,T_2](I[T_1,T_2](X,Y),Z) & \end{array}$$

KM-Monotonicity: If $X \leq Z$ and $Y \leq W$ then $\underline{x} \leq \underline{z}$, $\overline{x} \leq \overline{z}$, $\underline{y} \leq \underline{w}$ and $\overline{y} \leq \overline{w}$. So, $T_1(\underline{x},\underline{y}) \leq T_1(\underline{z},\underline{w})$ and $T_2(\overline{x},\overline{y}) \leq T_2(\overline{z},\overline{w})$. Thus, $[T_1(\underline{x},\underline{y}),T_2(\overline{x},\overline{y})] \leq [T_1(\underline{z},\underline{w}),T_2(\overline{z},\overline{w})]$ and therefore $I[T_1,T_2](X,Y) \leq I[T_1,T_2](Z,W)$.

Inclusion monotonicity: If $X \subseteq Z$ and $Y \subseteq W$, then $\underline{z} \leq \underline{x} \leq \overline{x} \leq \overline{z}$ and $\underline{w} \leq \underline{y} \leq \overline{y} \leq \overline{w}$. By definition, $I[T_1,T_2](X,Y) = [T_1(\underline{x},\underline{y}),T_2(\overline{x},\overline{y})]$ and $I[T_1,T_2](Z,W) = [T_1(\underline{z},\underline{w}),T_2(\overline{z},\overline{w})]$. Thus, by monotonicity of t-norms, $T_1(\underline{z},\underline{w}) \leq T_1(\underline{x},\underline{y}) \leq T_2(\overline{x},\overline{y}) \leq T_2(\overline{z},\overline{w})$. So, $[T_1(\underline{x},\underline{y}),T_2(\overline{x},\overline{y})] \subseteq [T_1(\underline{z},\underline{w}),T_2(\overline{z},\overline{w})]$. Therefore, $I[T_1,T_2](X,Y) \subseteq I[T_1,T_2](Z,W)$.

1-Identity: $\forall X \in I[0,1]$,

$$\begin{aligned} I[T_1,T_2](X,[1,1]) &= [T_1(\underline{x},1),T_2(\overline{x},1)] \\ &= [\underline{x},\overline{x}] \\ &= X. \end{aligned}$$

■

For simplicity, $I[T,T]$ will be denote by $I[T]$.

B. Obtaining pseudo t-norms from it-norms

In the following what will be shown is the inverse process, i.e. how to obtain two pseudo t-norms from an it-norm.

***Lemma** 5.1:* Let $\mathbb{T}$ be an it-norm. Then, the functions $\underline{\mathbb{T}} : [0,1] \times [0,1] \longrightarrow [0,1]$ and $\overline{\mathbb{T}} : [0,1] \times [0,1] \longrightarrow [0,1]$ defined by

$$\underline{\mathbb{T}}(x,y) = \pi_1(\mathbb{T}([x,x],[y,y]))$$

and

$$\overline{\mathbb{T}}(x,y) = \pi_2(\mathbb{T}([x,x],[y,y]))$$

are pseudo t-norms.

Proof: Clearly, $\underline{\mathbb{T}}$ and $\overline{\mathbb{T}}$ are well defined. In the following it will be proved that $\underline{\mathbb{T}}$ and $\overline{\mathbb{T}}$ are pseudo t-norms.

Symmetry: For each $x,y \in [0,1]$

$$\begin{aligned} \underline{\mathbb{T}}(x,y) &= \pi_1(\mathbb{T}([x,x],[y,y])) \\ &= \pi_1(\mathbb{T}([y,y],[x,x])) \\ &= \underline{\mathbb{T}}(y,x) \end{aligned}$$

Monotonicity: If $x_1 \leq x_2$ and $y_1 \leq y_2$ then, by definition of Kulisch-Miranker order, $[x_1,x_1] \leq [x_2,x_2]$ and $[y_1,y_1] \leq [y_2,y_2]$. So, by KM-monotonicity of $\mathbb{T}$, $\mathbb{T}([x_1,x_1],[y_1,y_1]) \leq \mathbb{T}([x_2,x_2],[y_2,y_2])$ and therefore, $\pi_1(\mathbb{T}([x_1,x_1],[y_1,y_1])) \leq \pi_1(\mathbb{T}([x_2,x_2],[y_2,y_2]))$. So, $\underline{\mathbb{T}}(x_1,y_1) \leq \underline{\mathbb{T}}(x_2,y_2)$.

1-identity: If $x \in [0,1]$, then $\underline{\mathbb{T}}(x,1) = \pi_1(\mathbb{T}([x,x],[1,1])) = \pi_1([x,x]) = x$.

So, $\underline{\mathbb{T}}$ is a pseudo t-norm. The case of $\overline{\mathbb{T}}$ is analogous one. ■

***Proposition** 5.2:* Let T_1 and T_2 be t-norms such that $T_1 \leq T_2$. Then $\underline{I[T_1,T_2]} = T_1$ and $\overline{I[T_1,T_2]} = T_2$.

Proof: Let $x,y \in [0,1]$. Then

$$\begin{aligned} \underline{I[T_1,T_2]}(x,y) &= \pi_1(I[T_1,T_2]([x,x],[y,y])) \\ &= \pi_1([T_1(x,y),T_2(x,y)]) \\ &= T_1(x,y) \end{aligned}$$

Analogously, $\overline{I[T_1,T_2]}(x,y) = T_2(x,y)$. ■

C. A partial order on it-norms

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be it-norms. Then, $\mathbb{T}_1 \leq \mathbb{T}_2$ if for each $X,Y \in \mathbb{I}$, $\mathbb{T}_1(X,Y) \leq \mathbb{T}_2(X,Y)$.

***Proposition** 5.3:* Let T_1 and T_2 be t-norms. If $T_1 \leq T_2$ then $I[T_1] \leq I[T_2]$.

Proof: Let $X,Y \in \mathbb{I}$. Then, by monotonicity of t-norms, $T_1(\underline{x},\underline{y}) \leq T_2(\underline{x},\underline{y})$ and $T_1(\overline{x},\overline{y}) \leq T_2(\overline{x},\overline{y})$. So, $[T_1(\underline{x},\underline{y}),T_1(\overline{x},\overline{y})] \leq [T_2(\underline{x},\underline{y}),T_2(\overline{x},\overline{y})]$. Therefore, $I[T_1] \leq I[T_2]$. ■

***Proposition** 5.4:* Let $\mathbb{T}$ be an it-norm. Then $I[T_W] \leq \mathbb{T} \leq I[T_G]$.

Proof: By lemma 5.1, $\underline{\mathbb{T}}$ and $\overline{\mathbb{T}}$ are pseudo t-norms. So, by proposition 4.1, $T_W \leq \underline{\mathbb{T}} \leq T_G$ and $T_W \leq \overline{\mathbb{T}} \leq T_G$. Since, trivially, $\underline{\mathbb{T}} \leq \overline{\mathbb{T}}$, then, for each $X,Y \in \mathbb{I}$, $[T_W(\underline{x},\underline{y}),T_W(\overline{x},\overline{y})] \leq [\underline{\mathbb{T}}(\underline{x},\underline{y}),\overline{\mathbb{T}}(\overline{x},\overline{y})] \leq [T_G(\underline{x},\underline{y}),T_G(\overline{x},\overline{y})]$. So, $I[T_W](X,Y) \leq \mathbb{T}(X,Y) \leq I[T_G](X,Y)$. ■

D. Representation theorem

In this subsection we will prove that the it-norm obtained as described in proposition 5.1 from a t-norm T is the best or optimal interval representation of T.

***Theorem** 5.1 (Representation theorem):* Let T be a t-norm. Then, $I[T]$ is the best interval representation of T, i.e. $I[T] = \widehat{T}$.

Proof: Since $\underline{x} \leq x \leq \overline{x}$ and $\underline{y} \leq y \leq \overline{y}$, for each $x \in X$ and $y \in Y$, then by monotonicity of T, $T(\underline{x},\underline{y}) \leq T(x,y) \leq T(\overline{x},\overline{y})$. So, $T(x,y) \in [T(\underline{x},\underline{y}),T(\overline{x},\overline{y})] = I[T](X,Y)$. On the other hand, since trivially $T(\underline{x},\underline{y}),T(\overline{x},\overline{y}) \in I[T](X,Y)$, then $I[T](X,Y)$ is the least interval containing $\{T(x,y) : x \in X \text{ and } y \in Y\}$. Therefore $I[T](X,Y) = \widehat{T}(X,Y)$. ■

***Proposition** 5.5:* Let T be a continuous t-norm. Then, for each $X,Y \in \mathbb{I}$, $I[T](X,Y) = \{T(x,y) : x \in X \text{ and } y \in Y\}$.

Proof: By theorem 5.1, $I[T](X,Y)$ is the least interval containing $\{T(x,y) : x \in X \text{ and } y \in Y\}$. Let $z \in I[T](X,Y)$. Then $T(\underline{x},\underline{y}) \leq z \leq T(\overline{x},\overline{y})$. So, by the well known middle value theorem, there exists $u,v \in [0,1]$ such that $\underline{x} \leq u \leq \overline{x}$ and $\underline{y} \leq v \leq \overline{y}$ and $T(u,v) = z$. Thus, $z = T(u,v) \in \{T(x,y) : x \in X \text{ and } y \in Y\}$. Therefore, $I[T](X,Y) = \{T(x,y) : x \in X \text{ and } y \in Y\}$. ■

E. Classes of it-norms

Analogously to the case of t-norms, several classes of it-norms can be defined. We only some analyze of them.

An it-norm $\mathbb{T}$ has **zero divisors** if there exists almost one pair of elements $X \neq [0,0]$ and $Y \neq [0,0]$, such that $\mathbb{T}(X,Y) = [0,0]$. For example, $I[T_W]([0.3,0.8],[0.5,0.6]) =$

$[0,0]$. So, if an it-norm does not have a zero divisor and $\mathbb{T}(X,Y)=0$ then $X=[0,0]$ or $Y=[0,0]$.

Let $\mathbb{T}$ be an it-norm. $\mathbb{T}$ is **Archimedean** if for each $X,Y \in \mathbb{I}-\{[0,0],[1,1]\}$, there exists a positive integer n such that $X^n < Y$ [2] where $X^1 = \mathbb{T}(X,X)$ and $X^{k+1}=\mathbb{T}(X,X^k)$.

Lemma *5.2:* Let T be a t-norm and $I[T]$ its associated it-norm. Then for each $X \in \mathbb{I}$ and positive integer n

$$X^n = [\underline{x}^n, \overline{x}^n].$$

Proof: If $n=1$ then

$$\begin{aligned} X^n &= I[T](X,X) \\ &= [T(\underline{x},\underline{x}), T(\overline{x},\overline{x})] \\ &= [x^n, x^n]. \end{aligned}$$

Suppose that for k, $X^k = [\underline{x}^k, \overline{x}^k]$. Thus, if $n=k+1$, then

$$\begin{aligned} X^n &= I[T](X,X^k) \\ &= I[T](X,[\underline{x}^k,\overline{x}^k]) \\ &= [T(\underline{x},\underline{x}^k), T(\overline{x},\overline{x}^k)] \\ &= [\underline{x}^n,\overline{x}^n]. \end{aligned}$$

■

An it-norm is **idempotent** if $\mathbb{T}(X,X)=X$ for each $X \in \mathbb{I}$, for example $I[T_G]$.

Theorem *5.2:* Let T be a t-norm. Then

1) T has zero divisors iff $I[T]$ has zero divisors.
2) T is Archimedean iff $I[T]$ is Archimedean.
3) T is continuous iff $I[T]$ is Scott-continuous iff is Moore-continuous.
4) T is idempotent iff $I[T]$ is idempotent.

Proof:

1) ($\Rightarrow$) If T has zero divisors x_0 and y_0 then $T(x_0,y_0)=0$. Thus, $[T(x_0,y_0),T(x_0,y_0)]=[0,0]$ and therefore, $I[T]([x_0,x_0],[y_0,y_0])=[0,0]$. So, $I[T]$ also has zero divisors.
($\Leftarrow$) If $I[T]$ has zero divisors X_0 and Y_0 then $I[T](X_0,Y_0)=[0,0]$. Thus, $[T(\underline{x_0},\underline{y_0}),T(\overline{x_0},\overline{y_0})]=[0,0]$ and therefore, $T(\overline{x_0},\overline{y_0})]=0$. Since, $X_0 \neq [0,0]$ and $Y_0 \neq [0,0]$ then $\overline{x_0} \neq 0$ and $\overline{y_0} \neq 0$. So, T also has zero divisors.
2) ($\Rightarrow$) Let $X,Y \in \mathbb{I}-\{[0,0],[1,1]\}$. If T is Archimedean then there exist positive integers n and m such that $\underline{x}^n < \underline{y}$ and $\overline{x}^m < \overline{y}$. If $n=1$ and $m=1$ then $T(\underline{x},\underline{x}) < \underline{y}$ and $T(\overline{x},\overline{x}) < \overline{y}$. Thus, $[T(\underline{x},\underline{x}),T(\overline{x},\overline{x})] < Y$. Therefore, $X^1 = I[T](X,X) < Y$. If $n=1$ and $m=k+1$ then $T(\underline{x},\underline{x}) < \underline{y}$ and $T(\overline{x},\overline{x}^k) < \overline{y}$. So, $I[T](X,[\underline{x},\overline{X^k}]) < Y$. But, by lemma 5.2 $\overline{X^k}=\overline{x}^k$ and as is well known $\underline{x}^k \leq \underline{x}$ and $\overline{x}^k \leq \overline{x}$. Therefore, $X^k \leq [\underline{x},\overline{X^k}]$. So, $I[T](X,X^k) < Y$ i.e. $X^{k+1} < Y$. By symmetry of it-norms and t-norms, if $n=k+1$ and $m=1$ then $X^{k+1} < Y$. If $n=k+1$ and $m=k'+1$ then $T(\underline{x},\underline{x}^k) < \underline{y}$ and $T(\overline{x},\overline{x}^k) < \overline{y}$. Therefore, by lemma 5.2 and definition of $I[T]$, $I[T](X,X^k) < Y$, i.e. $X^{k+1} < Y$.
($\Leftarrow$) Let $x,y \in (0,1)$. If $I[T]$ is Archimedean then there exists a positive integer n such that $[x,x]^n < [y,y]$. if $n=1$ then $I[T]([x,x],[x,x]) < [y,y]$ and therefore $[T(x,x),T(x,x)] < [y,y]$. So, $T(x,x) < y$, i.e. $x^1 < y$. If $n=k+1$ then $I[T]([x,x],[x,x]^k) < [y,y]$. Therefore, by lemma 5.2 and definition of $I[T]$, $[T(x,x^k),T(x,x^k)] < [y,y]$. So, $T(x,x^k) < y$ i.e. $x^{k+1} < y$.
3) Straightforward of theorem 3.1 which also is valid for functions from $\mathbb{I}\times\mathbb{I}$ into $\mathbb{I}$, as is the case of t-norms.
4) ($\Rightarrow$) If T is idempotent, then $T(x,x)=x$ for each $x \in [0,1]$. Thus, if $X \in \mathbb{I}$ then $[T(\underline{x},\underline{x}),T(\overline{x},\overline{x})]=[\underline{x},\overline{x}]=X$ and therefore, $I[T](X,X)=X$.
($\Leftarrow$) If $I[T]$ is idempotent, then $I[T](X,X)=X$ for each $X \in \mathbb{I}$. So, for each $x \in [0,1]$, $I[T]([x,x],[x,x]) = [T(x,x),T(x,x)]=[x,x]$ and therefore $T(x,x)=x$.

■

Since we can consider Scott as much as Moore continuity for it-tnorms, the interval extension for strict and nilpotent t-norms (which are based on continuity of t-norms) has two versions. A Scott-continuous Archimedean it-norm which has at least one pair of zero divisors is called **Scott-nilpotent** and is called **Scott-strict** otherwise. Analogously, for Moore continuity.

Corollary *5.1:* Let T be a t-norm. Then

1) T is nilpotent iff $I[T]$ is Scott-nilpotent iff $I[T]$ is Moore-nilpotent.
2) T is strict iff $I[T]$ is Scott-strict iff $I[T]$ is Moore-strict.

Proof: Straightforward of theorem 5.2 items 1, 2 and 3. ■

VI. Final Remarks

Since t-norms play a main role in fuzzy logics in the narrow sense, a "good´´ generalization of this concept is fundamental for the interval fuzzy logics in the narrow sense. In this way, this paper is a contribution in the consolidation of a formal study of interval fuzzy logics.

There are in the literature other attempts to extend the t-norm notion for intervals. For example, in [41] it was demanded that beyond the condition imposed in our generalization also be continuous w.r.t. Moore topology. This exigency is hard because not all t-norms will have an interval t-norm which represents them. In fact Zuo neither provides any relation between t-norms and interval t-norms (in the sense of providing a way to obtain canonically t-norms from interval t-norms and vice-versa) nor considers the representation aspects of interval t-norms and therefore can not be useful for the fourth approach pointed out by Lodwick in [26]. On the other hand, in [11] and [12] it was required that for each $X \in \mathbb{I}$, $\mathbb{T}([0,1],X)=[0,\overline{x}]$, that $\mathbb{T}$ be distributive under $\wedge$ and $\vee$, that $\mathbb{T}([x,x],[x,x])$ return a degenerated interval and is not required be monotonic (neither inclusion nor Kulisch-Miranker). An interesting result in [11] is the theorem 7 where it is proved that for each it-norm $\mathbb{T}$ (in their sense) there exists a t-norm T such that

$$\mathbb{T}(X,Y) = [T(\underline{x},\underline{y}), T(\overline{x},\overline{y})] \tag{1}$$

So, each it-norm has associated an unique t-norm (clearly $\mathbb{T}$ represents T). That is, if $\mathbb{T}=\mathbb{T}'$ then $T=T'$. On the

[2] $X < Y$ iff $X \leq Y$ and $X \neq Y$

other hand this theorem does not guarantees that each t-norm has an associated it-norm satisfying (1). For example, suppose that the Lukasiewics t-norm T_L has an associated it-norm $\mathbb{T}_L$. Therefore

$$\mathbb{T}_L(X,Y) = [T_L(\underline{x},\underline{y}), T_L(\overline{x},\overline{y})]$$

Thus,

$$\begin{aligned} \mathbb{T}_L([0.6,0.7],[0.2,0.8] \vee [0.3,0.5]) &= \\ \mathbb{T}_L([0.6,0.7],[0.3,0.8]) &= \\ [T_L(0.6,0.3), T_L(0.7,0.8)] &= [0,0.5]. \end{aligned}$$

On the other hand, by distributivity of it-norms of [12], $\mathbb{T}_L([0.6,0.7],[0.2,0.8] \vee [0.3,0.5]) = \mathbb{T}_L([0.6,0.7] \vee [0.2,0.8],[0.6,0.7] \vee [0.3,0.5])$. But

$$\begin{aligned} \mathbb{T}_L([0.6,0.7] \vee [0.2,0.8],[0.6,0.7] \vee [0.3,0.5]) &= \\ \mathbb{T}_L([0.6,0.8],[0.6,0.7]) &= \\ [T_L(0.6,0.6), T_L(0.8,0.7)] &= [0.2,0.5] \end{aligned}$$

generating an inconsistency. Therefore, there does not exist an it-norm associated to T_L!!!.

Thus, the main contribution of our interval generalization of t-norm, w.r.t. other generalizations, is to regard interval t-norms as interval representations of t-norms. This vision agrees with the interval fuzzy approach where the intervals degree membership is considered as an imprecision in the degree of belief of a specialist, i.e. as a representation or an approximation of the exact degree. Other approaches do not consider this factor.

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