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THE INTERVAL CONSTRUCTOR ON CLASSES OF ML-ALGEBRAS

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Dissertation presented to the Program of Post Graduation in Systems and Computation of Department of Informatics and Applied Mathematics of Federal University of Rio Grande do Norte required to obtain the degree of Master in Systems and Computation (MSc.).

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For my parents...

my mentors and heroes

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ABSTRACT

Monoidal logic, ML for short, which formalized the fuzzy logics of continuous t-norms and their residua, has arisen great interest, since it has been applied to fuzzy mathematics, artificial intelligence, and other areas. It is clear that fuzzy logics basically try to represent imperfect or fuzzy information aiming to model the natural human reasoning. On the other hand, in order to deal with imprecision in the computational representation of real numbers, the use of intervals have been proposed, as it can guarantee that the results of numerical computation are in a bounded interval, controlling, in this way, the numerical errors produced by successive roundings. There are several ways to connect both areas; the most usual one is to consider interval membership degrees. The algebraic counterpart of ML is ML-algebra, an interesting structure due to the fact that by adding some properties it is possible to reach different classes of residuated lattices. We propose to apply an interval constructor to ML-algebras and some of their subclasses, to verify some properties within these algebras, in addition to the analysis of the algebraic aspects of them.

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Area of Concentration: Fuzzy Logic
Key words: Fuzzy logics; Monoidal logic; Interval mathematics; T-norm; Residuated lattices; ML-algebras

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Chapter 1

Introduction

Everyday natural language contains a lot of inexact and vague information, or in other words, fuzzy information. In order to represent and manipulate this kind of data, we need more than the classical logic, which only admits truth or falseness. The mathematical modeling of fuzzy concepts was presented by Lofti Zadeh in [Zad65] considering that meaning, in natural language, was a matter of degree. A proposition was no longer simply true or false; instead, a real value in the interval $[0, 1]$ was considered to indicate how much that proposition was believed to be true.

This led to the development of many studies concerning the fuzzy logics. In the domain of engineering and applied sciences, for instance, they have been studied as a tool to deal with the natural uncertainty of knowledge and to represent the uncertainty of human reasoning. In this way, fuzzy systems try to reason in a similar way human beings do by attempting to transform vague information into adequate solutions to given problems or by making decisions and taking actions using this imprecise knowledge. In a different direction of study, fuzzy logics, in a certain way, enrich classical logic. They act as a symbolic logic with a comparative notion of truth, having as a base the classical logic, once the connectives must behave as in the classical logic on the extremes values (0 and 1). Some examples of fuzzy logics are: Monoidal Logic (ML, for short) introduced by Höhle in [Höh95]; Monoidal t-norm based Logic (MTL) introduced by Esteva and Godo in [EG01]; Basic Logic, (BL, presented by Hájek in [Háj98]); Gödel Logic (G, in [Göd32]); and Łukasiewicz Logic (Ł, in [ŁT30]).

In systems with the approach that not only the available information is vague or uncertain, but also the membership degrees, it is common to use the interval fuzzy logics. Once the membership degrees are uncertain, they are represented by intervals in the unit interval $[0, 1]$. Interval-valued fuzzy sets were introduced independently by Zadeh [Zad75] and others authors (e.g., [GG75], [Jah75], [Sam75]) in the 70's. The connection between the fuzzy theory and interval mathematics has been studied concerning different points of view like, for example, in [CDK06], [DP91], [DP05], [GWW96], [KY95], [KM00], [ML03], [NW00], [NKZ97], [Tur86] and [YMK99]. And a good contribution for the formal study of interval fuzzy logics was done by Bedregal et. al. in [BT06] where an interval t-norm was seen as an interval representation of t-norms in the sense of [SBA06]. This work was extended in [BSCB06] for the bounded lattice context. Moreover, from a categorical point of view, it was proved in [BCBS07] the agreement between the interval generalization of automorphisms and this t-norm generalization, in the sense that the interval constructor of automorphisms and t-norms could be seen as a functor preserving the action of automorphisms on t-norms. Thus, it was obtained the best interval representation (in the sense of [BT06]) of any t-norm and automorphism leading to handle with the optimality of interval fuzzy algorithms.

Recently, fuzzy logics based on t-norms and their residua have been widely investigated. In particular, Höhle's Monoidal Logic (ML), whose algebraic counterpart is the complete class of residuated lattices (namely a ML-algebra), seems quite interesting due to the possibility of reaching other logics just by adding some axioms. Moreover, it is also possible to add properties to ML-algebras and get different classes of residuated lattices, i.e. a prelinear ML-algebra is a MTL-algebra.

In this way, our work seeks to introduce the interval constructor to ML-algebras and some of their subclasses. Besides, it verifies some properties within these algebras, in addition to the analysis of the algebraic aspects of them. Our motivation has emerged from some related works that will be discussed in the following section.

1.1 Motivation and Related Works

Much work had already been done within interval fuzzy logics. The interval-valued fuzzy set theory (introduced independently in the mid-seventies by Grattan-Guinness [GG75], Jahn [Jah75], Sambuc [Sam75] and Zadeh [Zad75]) is an increasingly popular extension of fuzzy set theory where traditional $[0,1]$-valued membership degrees were substituted by intervals in the unit interval $[0,1]$, attempting to approximate the imprecise degrees.

In this way, not only vagueness of information was being dealt with, but also a feature of uncertainty on the membership degree, that could be addressed intuitively through the intervals. Interval-valued fuzzy sets (IVFSs) are considered to have a great potential, especially to approximate reasoning applications. However, a formal treatment of interval fuzzy logics is still lacking. According to [CDK06], two main reasons for that can be stressed: an underestimation of the richness of the associated algebraic structure and a certain preference for prelinear algebraic structures.

Nowadays, there is an important research field on fuzzy logics focused on prelinear residuated structures. And we can state that the algebraic counterpart of the logics mentioned above (ML, MTL, BL, G, Ł) are bounded lattices, which guarantee a more general view. Then, the partial ordering of a bounded lattice $\mathcal{L}$ serves to compare the truth values of formulas which can be true to some extent. And considering the examples given, all those logics presuppose the prelinearity property, which is held in every residuated lattice. However, when we deal with closed intervals of a bounded lattice, this property is not necessarily preserved [GCDK07].

Van Gasse et al., in [GCDK07], presented Triangle Logic (TL), a system of many-valued logic which captured the tautologies of interval-valued residuated lattices (IVRLs). To do so, they also introduced triangle algebras, proving soundness and completeness of TL concerning those algebras. The importance of their work lies in the axiomatic formalization of residuated t-norm based logics on the lattice of closed intervals of $[0,1]$.

Our work differs from the investigations shown above in the sense that we are using the interval constructor on some classes of ML-algebras and, based on the results, we are going to add properties which end up on very known extensions of ML-algebras.

In order to fulfill the motivations presented above we have studied fuzzy logics and proposed some generalizations of t-norms.

We had introduced, for instance, one extension of t-norms to arbitrary complete lattices in [BS05]. That work was a monograph presented to the Federal University of Rio Grande do Norte to obtain the degree of Bachelor in Computer Sciences. We had showed that the classic logical connectives, some t-norms and interval t-norms, and the triangular versions of the other connectives were functions on specific lattices satisfying certain properties. Thus, we could generalize the propositional classic connectives, as well as the fuzzy and interval fuzzy connectives using the lattice theory, which is a more general concept.

Since then, some other papers were published to carry on that study. For instance, in [BSCB06], we considered a well known generalization of the t-norm notion for arbitrary bounded lattices and introduced two generalizations of the automorphism notion for arbitrary bounded lattices. We also showed some basic constructions on those notions, based on similar constructions for lattices. Besides, we could see that although the category introduced there was not Cartesian, it had several properties similar to the interval category of [CBB01] and, considering an interval constructor based on the Scott partial order [Sco70], that category was also similar to an Acióly-Scott interval category in [CBB04].

Moreover, in the introductory paper entitled "Bounded Lattice T-Norms as an Interval Category", [BCBS07], we also considered a well known generalization of the t-norm notion for arbitrary bounded lattices. We introduced the t-norm morphism, which is a generalization of the automorphism notion for t-norms on arbitrary bounded lattices. Those two generalizations were considered a rich category having t-norms as objects and t-norm morphism as morphism. We proved that putting together that category, a natural transformation introduced there, and the interval constructor on bounded lattice t-norms and t-norm morphisms resulted in an interval category in the sense of [CBB01].

Finally, the same generalization mentioned above (done in [BSCB06] and [BCBS07]) was also taken into account in [BSCB07]. However, in this paper we proved that such category was Cartesian and whenever its subcategory whose objects were strict t-norms, we proved that it was a Cartesian closed category. Moreover, we showed that the usual interval construction on lattices was a functor on those categories.

Since t-norms play an important role in fuzzy logics, a good generalization of them is extremely important in interval fuzzy logics. In [BT06], an interval t-norm was seen as an interval representation of t-norms according to the interval fuzzy approach where the interval membership degree was considered inaccurate in the belief degree of an expert, or in other words, a representation or an approximation of the exact degree. Even though, other approaches took no account of that fact. In addition, from a categorical point of view, it was proved that the interval generalization of automorphisms agrees with that generalization of t-norms, in the sense that the interval constructor of automorphisms and t-norms could be seen as a functor preserving the action of automorphisms on t-norms. Therefore, the best representation of any t-norm and automorphism was achieved, which means that it can be used to deal with optimal interval fuzzy algorithms.

Considering the areas of study that have just been mentioned, the main purpose of this work is to continue part of the investigations shown above. As stated before, we intend to use the interval constructor on classes of ML algebras verifying some properties which still hold. Besides, we are also willing to extend the ML logic considering the interval constructor.

1.2 Chapters Summary

This document is divided into six chapters, where the first one was a brief outline of our work. The other chapters proceed as follows: chapter 2 is a collection of definitions used, explaining some basic, but important concepts which will be used throughout our work. Advanced readers should skip to chapter 3, where the fuzzy connectives on lattices are presented, as well as the interval constructor. The forth chapter discusses a little bit of ML logic and its extensions and ML-algebras. Finally, our proposal of applying the interval constructor on ML-algebras and some of their subclasses can be found in the fifth chapter. The sixth, and last, chapter contains the concluding remarks and possible future works.

Chapter 2

Background

The idea of doing a self contained document requires a chapter explaining some basic concepts. Advanced readers should skip to the following chapter.

2.1 Fuzzy Logics

Fuzzy set theory was first introduced by Lofti Zadeh in [Zad65]. The initial idea was to consider a belief degree, i.e. a real value in the interval $[0, 1]$, which indicated how much a given element was believed to belong to a set. Then, each element of the fuzzy set (represented here by X) was mapped to $[0, 1]$ by a membership function $\mu : X \rightarrow [0, 1]$, where $[0, 1]$ meant real numbers between 0 and 1. Otherwise, classical logic only admits true or false statements, not both. So, if you are tall in the classical logic, you can not belong to the group of people who are short. However, in fuzzy logics if your height is considered not so tall, you can belong to the group of short people with a certain degree of membership. Thus, this theory is appropriate to deal with non-precise information. The subjacent logic to the fuzzy set theory is named fuzzy logic, or fuzzy logics, since there has been a great variety of interpretations and studies underneath this logic. However, it is important to bear in mind that fuzzy logics constitute an important tool to deal with the natural uncertainty of knowledge and to represent the uncertainty of human reasoning. They guarantee that even having uncertain data, we can use them to infer conclusions and convey solutions to our problems. Fuzzy systems can be said to reason in a similar way human beings do because human reasoning also involves inference mechanisms to convert available information into adequate solutions. It is also allowed to say that fuzzy logics generalize the classical logic, despite using intermediate values of truth, since they behave exactly in the same way on the extreme values 0 and 1.

Fuzzy logics can be distinguished in two main directions: in a broader sense they can be considered an apparatus to analyze uncertain notions, for instance, in natural languages and control systems or, in other words, their main goal is to develop computational systems based on fuzzy reasoning. In a narrower sense, they act as a symbolic logic with a comparative notion of truth, having as a base the classical logic, once the connectives must behave as in the classical logic on the extreme values (0 and 1).

2.1.1 T-norms, T-conorms and Other Connectives

Willing to model probabilistic metric spaces, Menger introduced the notion of triangular norms, t-norms in short, in [Men42]. However, the axiomatic definition for t-norms was only proposed in [SS61] by Schweizer and Sklar in 1961 and the dual notion of the t-norm, called t-conorm, appeared with the work of Alsina, Trillas and Valverde in [ATV80]. In that work, t-norms and t-conorms were used to model the conjunction and disjunction connectives in fuzzy logics, generalizing several definitions for fuzzy logics provided by Zadeh in [Zad65]. Other usual connectives can be fuzzy extended as well from a t-norm or defined after an independent given axiomatization. In [BC06], the fuzzy extension of propositional connectives was called propositional fuzzy semantics, or simply, fuzzy semantics. Thus, each fuzzy semantics can determine a different set of true formulas (1-tautologies) and false formulas (0-contradictions) and, therefore, different (fuzzy) logics. For example, it is possible to obtain a fuzzy canonical interpretation for the implication and negation connectives from a t-norm. Each t-norm determines a different set of tautologies and contradictions, and consequently, distinct fuzzy logics [BT06].

The following topics present the general definitions of t-norm, t-conorm, implication and negation, taken from [Met05] and [NW00].

2.1.1.1 T-norms

A t-norm is a binary operation, generally used to represent the conjunction, or the intersection. It was defined as a function $T : [0,1] \times [0,1] \to [0,1]$ (or $T : [0,1]^2 \to [0,1]$) such that, for all $x, y, z \in [0,1]$, the following properties are satisfied:

1. Commutativity: $T(x, y) = T(y, x)$;
2. Associativity: $T(x, T(y, z)) = T(T(x, y), z)$;
3. Monotonicity: $x \leq y$ implies $T(x, z) \leq T(y, z)$;
4. Identity: $T(x, 1) = x$.

Note that we are using the prefix notation $T(x, y)$ for the t-norms, but the infix notation $(x * y)$ is often used in the literature to emphasize the interpretation of a logical connective. We will freely use both notations during this work and, eventually, the $*$ symbol may be substituted by a different symbol.

There are several (in fact, uncountable) types of t-norms, each one having certain properties and usually arranged into families. And this variety of interpretations leads to different classes of logics. Some properties of t-norms are:

Definition 2.1 *Idempotency : $T : [0,1]^2 \to [0,1]$ is idempotent iff $T(x, x) = x$, for all $x \in [0,1]$.*

That property means if we repeat something, it doesn't make it more or less true.

Definition 2.2 *Continuity :* $T : [0,1]^2 \rightarrow [0,1]$ *is continuous iff for all* $x, y \in [0,1]$*, given a sequence* $(x_i)_{i \geq 0}$*,* $x_i \in [0,1]$ *such that* $x = \lim_{i \to \infty} x_i$*, then also* $lim_{i \to \infty} T(x_i, y) = T(x, y)$*.*

Continuity ensures the value of the function is not too sensitive to changes in the values of its arguments. Besides, there is also the property of left-continuity, defined below:

Definition 2.3 *Left-Continuity :* $T : [0,1]^2 \rightarrow [0,1]$ *is left-continuous iff for all* $x, y \in [0,1]$*, given a sequence* $(x_i)_{i \geq 0}, x > x_i \in [0,1]$ *such that* $x = \lim_{i \to \infty} x_i$*, then also* $\lim_{i \to \infty} T(x_i, y) = T(x, y)$*.*

Some of the fundamental (continuous) t-norms are ([Met05]):

- Gödel t-norm : $G(x, y) = min(x, y)$;
- Product t-norm: $P(x, y) = x \cdot y$;
- Łukasiewicz t-norm: Ł $= max(0, x + y - 1)$.

Other non-continuous t-norms:

- Weak t-norm (W), where:

$$W(x,y) = \begin{cases} \min\{x,y\} & \text{, if } max\{x,y\} = 1 \\ 0 & \text{, otherwise.} \end{cases} \tag{2.1}$$

- Nilpotent minimum t-norm (N):

$$N(x,y) = \begin{cases} \min\{x,y\} & \text{, if } x + y > 1 \\ 0 & \text{, otherwise.} \end{cases} \tag{2.2}$$

For two given t-norms, T_1 and T_2, we can define if one is stronger or weaker than the other one. We say that T_1 is weaker than T_2 (or, equivalently, T_2 is stronger than T_1), written $T_1 \leq T_2$, if $T_1(x, y) \leq T_2(x, y)$, for all $x, y \in [0, 1]$. An interesting result is that:

Proposition 2.1 $W \leq T \leq G$ *, for any t-norm* T*.*

Proof: By the monotonicity and commutativity properties, we have: $T(x,y) \leq T(x,1) \leq x$, $T(x,y) = T(y,x) \leq T(y,1) \leq y$. This means that: $T(x,y) \leq min(x,y)$. Thus, $T \leq G$. On the other hand, if $max(x,y) \neq 1$, then $W(x,y) = 0$ and so, $W(x,y) \leq T(x,y)$. If $max(x,y) = 1$, then $x = 1$ or $y = 1$. Suppose $x = 1$ (analogous for the case where $y = 1$), then $W(x,y) = min(x,y) = y = T(y,1) = T(y,x) = T(x,y)$. Consequently, $W(x,y) \leq T(x,y)$. ■

2.1.1.2 T-conorms

A triangular conorm, t-conorm in short, is used to represent the disjunction. It was defined as a function $S : [0,1]^2 \to [0,1]$, such that, for all $x, y, z \in [0,1]$, the properties below are held:

1. Commutativity: $S(x,y) = S(y,x)$;
2. Associativity: $S(x, S(y,z)) = S(S(x,y), z)$;
3. Monotonicity: $x \leq y$ implies $S(x,z) \leq S(y,z)$;
4. Identity: $S(x,0) = x$.

There is also the infix notation for the t-conorm, that would be $x \oplus y$. However, the symbol can vary a lot in the literature, and it will be specified along this work, whenever necessary.

A t-norm can be used to define a t-conorm and vice versa. So, we can call S_T the dual t-conorm of the t-norm T , where $S_T(x,y) = 1 - T(1-x, 1-y)$.

In this way, the dual t-conorms of the fundamental t-norms are as showed below ([Met05]):

- Maximum t-conorm: $S_G(x,y) = max(x,y)$;

- Probabilistic sum t-conorm: $S_P(x,y) = x + y - x \cdot y$;
- Łukasiewicz t-conorm: $S_Ł(x,y) = min(1, x+y)$.

And the dual notion of the weak t-norm is:

- Strong t-conorm:

$$S_W(x,y) = \begin{cases} \max\{x,y\} & \text{, if } \min\{x,y\} = 0 \\ 1 & \text{, otherwise.} \end{cases}$$

Analogously, for two given t-conorms, S_1 and S_2, we can define if one is stronger or weaker than the other one. We say that S_1 is weaker than S_2 (or, equivalently, S_2 is stronger than S_1), written $S_1 \leq S_2$, if $S_1(x,y) \leq S_2(x,y)$, for all $x, y \in [0,1]$. And it is also easily proved that $S_G \leq S \leq S_W$, for any t-conorm S.

2.1.1.3 Implication

In the literature, we can find many different definitions of fuzzy implications, as well as several representations of the infix and prefix notations (which will be freely used throughout this work). A lot of properties for implication can also be required to achieve $x \Rightarrow y$ or $P(x,y)$. For instance, Fodor and Roubens in [FR94] define a fuzzy implication as a function $P : [0,1]^2 \to [0,1]$, which satisfies the following properties:

- $x \leq z$ implies $P(x,y) \geq P(z,y)$, for all $x, y, z \in [0,1]$;
- $y \leq z$ implies $P(x,y) \leq P(x,z)$, for all $x, y, z \in [0,1]$;
- $P(0,y) = 1$, for all $y \in [0,1]$;
- $P(x,1) = 1$, for all $x \in [0,1]$;
- $P(1,0) = 0$.

Nevertheless, it seems reasonable to seek an implication which can be connected to our notion of conjunction, obtained as a dual notion of the t-norm T.

Definition 2.4 *Let T be a t-norm. Then, $I_T(x, y) = sup\{z \in [0, 1] : T(x, z) \leq y\}$, for all $x, y \in [0, 1]$, defines a fuzzy implication called **R-implication** or residuum of T.*

The name R-implication comes from the term residual implication, once this implication is related to a residuation concept from the Intuitionistic logic. Actually, it has been shown that, in this context, this name is proper only for left-continuous t-norms. The acronym *sup* is the *supremum*, which will be explained on the following chapter.

It is important to emphasize that an R-implication is well defined only if the t-norm is left-continuous, that is, if T satisfies the residuation condition shown below:

$$T(x, y) \leq z \text{ iff } y \leq I_T(x, z), \text{ for all } x, y, z \in [0, 1].$$

We usually say that T and I_T form and adjoint pair , represented by (T, I_T).

According to [BJ07],[FR94], among others, if T is any t-norm (not necessarily left-continuous), then I_T satisfies the following properties:

i. $I_T(1, y) = y$, $y \in [0, 1]$ (left neutrality property);

ii. $I_T(x, x) = 1$, $x \in [0, 1]$ (identity principle);

Besides, if I_T is an R-implication based on a left-continuous t-norm, then it satisfies i., ii. and these two other properties:

iii. $I_T(x, I_T(y, z)) = I_T(y, I_T(x, z))$, $x, y, z \in [0, 1]$ (exchange principle);

iv. $x \leq y$ iff $I_T(x, y) = 1$, $x, y \in [0, 1]$ (ordering property).

The *residua* for the fundamental t-norms are [Met05]:

- Gödel implication:
$$P_G(x,y) = \begin{cases} 1 & \text{, if } x \leq y \\ y & \text{, otherwise.} \end{cases}$$

- Product implication:
$$P_{\prod}(x,y) = \begin{cases} 1 & \text{, if } x \leq y \\ \frac{y}{x} & \text{, otherwise.} \end{cases}$$

- Łukasiewicz implication: $P_{Ł}(x,y) = min(1, 1 - x + y)$.

2.1.1.4 Bi-implication

A fuzzy bi-implication was defined in [BC06] as follows. A function $B : [0,1]^2 \rightarrow [0,1]$ is a bi-implication, if it satisfies the following properties, for all $x, y, z \in [0,1]$:

- $B(x,y) = B(y,x)$;
- If $x = y$, then $B(x,y) = 1$;
- $B(0,1) = 0$;
- If $x \leq y \leq z$, then $B(x,y) \geq B(x,z)$ and $B(y,z) \geq B(x,z)$.

2.1.1.5 Negation

A function $N : [0,1] \rightarrow [0,1]$ can model a negation if it is non-increasing (if $x \geq y$, then $N(x) \leq N(y)$, for all $x, y \in [0,1]$), $N(0) = 1$ and $N(1) = 0$ [Met05].

We call a negation function **strict** if, and only if, it is strictly decreasing and continuous. N is **strong** if, and only if, it is an involution i.e.: $N(N(x)) = x$, for all $y \in [0,1]$. Observe that strong negations are strict. If N is not strong, then N is called **weak**.

2.2 Groups

A **group** G is an algebra $\langle G, *, \ ^{-1}, 1\rangle$ with a binary, a unary, and nullary operation, in which the following properties hold:

- G_1: $x * (y * z) = (x * y) * z$;
- G_2: $x * 1 = 1 * x = x$;
- G_3: $x * x^{-1} = x^{-1} * x = 1$.

A group G is Abelian (or commutative) if, in addition, the forth property is satisfied:

- G_4: $x * y = y * x$.

A **semigroup** is an algebraic structure on a set, with a binary operator $\langle G, *\rangle$, in which the multiplication operation $(*)$ is associative. And, a **monoid** is an algebra $\langle M, *, 1\rangle$, with a binary operation satisfying G_1 and G_2. Or, on the other hand, a monoid can also be thought as a semigroup with an identity element.

A commutative (or Abelian) semigroup is defined in [Háj98] as follows:

Definition 2.5 *A semigroup $\langle G, *\rangle$ is commutative if it satisfies the axiom of commutativity: $x * y = y * x$ (for all x, y).*

The concepts explained above are somehow linked to the following chapter, where the fuzzy connectives on lattices are introduced, as well as the interval constructor.

Chapter 3

Fuzzy Connectives on Lattices and the Interval Constructor

The lattice theory is the study of sets of objects called lattices. Although the origin of the lattice concept can be traced back to Boole's analysis of thought, the study of it was given a great boost by a series of papers and a textbook [Bir67] written by Birkhoff.

3.1 The Lattice Theory

We can define lattice in two standard ways: as an algebraic structure or as a partial ordered set. However, we can say that there is a biunivocal correspondence between them (for more details refer to [BS81], [Joh82]). Both constructions can be considered equivalent and whenever we mention the word lattice throughout the text, it will mean, in general, lattice by the first definition. But we can also use the second definition according to its convenience.

Concerning the algebraic definition, a nonempty set L can be defined as a lattice together with two operations $\vee$ and $\wedge$ named **join** and **meet**, respectively, if it satisfies the following identities:

- L_1 Commutativity: $x \wedge y = y \wedge x$, and $x \vee y = y \vee x$;
- L_2 Associativity: $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, and $x \vee (y \vee z) = (x \vee y) \vee z$;
- L_3 Idempotency: $x \wedge x = x$, and $x \vee x = x$;
- L_4 Absorption: $x \wedge (x \vee y) = x$, and $x \vee (x \wedge y) = x$.

An interesting example can be given if we let L be a set of propositions, $\wedge$ denote the conjunction and $\vee$ denote the disjunction. It's well known that L_1 to L_4 are properties from propositional logic which still hold in a lattice framework.

Explaining the other definition of a lattice requires a bit understanding of a partial order on a set. Thus, we say that a binary relation $\leq$ on a set A is a partial order on the set A if the reflexivity, anti-symmetry and transitivity conditions hold, where:

- Reflexivity means: $a \leq a$;
- Anti-symmetry: $a \leq b$ and $b \leq a$ imply $a = b$; and,
- Transitivity: $a \leq b$ and $b \leq c$ imply $a \leq c$.

We say $\leq$ is a total order on the set A if, in addition, for every $a, b \in A$, it satisfies either $a \leq b$ or $b \leq a$.

So, a nonempty set, with a partial order on it, is called a partially ordered set, **poset** for short, and if the relation is a total order, then we call it a totally ordered set, or a linearly ordered set, or simply a **chain**.

Now, let A be a subset of a poset P. We say an element p, in P, is an upper bound for A if $a \leq p$, for every $a \in A$. Besides, we say that p is the least upper bound of A, or *supremum* of A ($sup\ A$), if p is an upper bound of A, and if b is another upper bound of A, then $p \leq b$ (in other words, p is the smallest element among the upper bounds of A). Similarly, we can understand the *infimum* as the greatest lower bound of A.

A poset L is an ordered lattice if, and only if, for every a, b in L, both the *supremum* and *infimum* exist. Each ordered lattice determines a lattice and vice versa.

Proposition 3.1 *Let $\mathcal{L} = \langle L, \leq \rangle$ be an ordered lattice. Then, $\mathcal{L} = \langle L, \wedge, \vee \rangle$, where $x \wedge y = inf\{x, y\}$ and $x \vee y = sup\{x, y\}$, is a lattice.*

Proof: See [Joh82]. ■

Examples: $B = \langle \{0, 1\}, \leq \rangle$, $F = \langle [0, 1], \leq \rangle$ and $\mathbb{I}F = \langle I[0, 1], \leq_{KM} \rangle$ are lattices, where $I[0, 1] = \{[a, b] : 0 \leq a \leq b \leq 1\}$ and $[a, b] \leq_{KM} [c, d]$ iff $a \leq c$ and $b \leq d$ (note that $\leq_{KM}$ is the Kulisch-Miranker order).

The algebraic lattices associated to B, F and $\mathbb{I}F$ are $\langle \{0, 1\}, \wedge_B, \vee_B \rangle$, $\langle [0, 1], \wedge_F, \vee_F \rangle$ and $\langle I[0, 1], \wedge_{\mathbb{I}F}, \vee_{\mathbb{I}F} \rangle$, respectively, where $\wedge_B$ and $\vee_B$ are the classical conjunction and disjunction, respectively; $\wedge_F$ and $\vee_F$ are Gödel t-norm and t-conorm, respectively; and, finally, $\wedge_{\mathbb{I}F}$ and $\vee_{\mathbb{I}F}$ are defined by: $[a, b] \wedge_{\mathbb{I}F} [c, d] = [a \wedge_F c, b \wedge_F d]$ and $[a, b] \vee_{\mathbb{I}F} [c, d] = [a \vee_F c, b \vee_F d]$.

We can define an inherent order in a lattice $\mathcal{L} = \langle L, \wedge, \vee \rangle$, represented by $\leq_{\mathcal{L}}$. So, for all $x, y \in L$, $x \leq_{\mathcal{L}} y$ iff $x \wedge y = x$. Or, analogously, $x \leq_{\mathcal{L}} y$ iff $x \vee y = y$.

Proposition 3.2 *If $\mathcal{L} = \langle L, \wedge, \vee \rangle$ is a lattice, then $\langle L, \leq_{\mathcal{L}} \rangle$ is an ordered lattice, where $x \leq_{\mathcal{L}} y$ iff $x = x \wedge y$ (or $y = x \vee y$).*

Proof: See [Joh82]. ■

Observe that if $\langle L, \leq \rangle$ is an ordered lattice, then the lattice, obtained as in proposition 3.1, has as an inherent order ($\leq_{\mathcal{L}}$) the $\leq$ order itself. That is, if we apply proposition 3.1 to an ordered lattice $\langle L, \leq \rangle$ and then, if we apply proposition 3.2 to the lattice obtained in that way, we will get again the same ordered lattice $\langle L, \leq \rangle$. Similarly, if $\mathcal{L} = \langle L, \wedge, \vee \rangle$ is a lattice, then, after applying proposition 3.2, we have an ordered lattice $\langle L, \leq_{\mathcal{L}} \rangle$, which after applying proposition 3.1, result in the same original lattice. With these results we can highlight the correspondence between a lattice and an ordered lattice, justifying the treatment that a lattice and an ordered have the same structure. Therefore, from now on, we will call both as lattices.

3.1.1 Classes of Lattices

A **bounded lattice** is an algebraic structure $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ such that $\langle L, \wedge, \vee\rangle$ is a lattice, and the constants 0 and $1 \in L$ satisfy the following properties:

- For all $x \in L$, $x \wedge 1 = x$ and $x \vee 1 = 1$;
- For all $x \in L$, $x \wedge 0 = 0$ and $x \vee 0 = x$.

The constant 1 is called upper bound, or **top** of $\mathcal{L}$, and 0 is the lower bound, or **bottom** of $\mathcal{L}$.

Some examples:

1. $\mathcal{L}_T = \langle \{1\}, \wedge, \vee, 1, 1\rangle$, where $1 \wedge 1 = 1 \vee 1 = 1$.
2. $\mathcal{B} = \langle B, \wedge, \vee, 0, 1\rangle$, where $B = \{0, 1\}$ and $\wedge$, $\vee$ are as in the Boolean algebra.
3. $\mathbb{I} = \langle [0,1], \wedge, \vee, 0, 1\rangle$, where $x \wedge y = min(x, y)$ and $x \vee y = max(x, y)$.
4. $\mathbb{I}_{\mathbb{Q}} = \langle [0,1] \cap \mathbb{Q}, \wedge, \vee, 0, 1\rangle$, where $\mathbb{Q}$ is the set of rational numbers, $x \wedge y = min(x, y)$ and $x \vee y = max(x, y)$.

A bounded lattice $\mathcal{L}$ is said **complete** if every subset $\mathcal{M}$ of $\mathcal{L}$ has both a least upper bound and a greatest lower bound. In other words, a poset $\mathcal{L}$ is complete if both *supremum* and *infimum* of $\mathcal{L}$ exist for each subset.

Note that all complete lattices are bounded (where the constant 0 equals the *supremum* on the empty set and 1 equals the *infimum* on a given set $\mathcal{L}$), but not all bounded lattices are complete, for example, the set $\mathbb{I}_{\mathbb{Q}}$, since the set $\{x \in [0,1] \cap \mathbb{Q} : x < 2\sqrt[n]{2} - 1\}$ does not have the *supremum* in $\mathbb{Q}$.

The algebraic structure $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is named **residuated lattice** if:

i. $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice;

ii. $\langle L, *, 1\rangle$ is a commutative monoid;

iii. $\Rightarrow$ is a function $L^2 \rightarrow L$ such that, for all $x, y, z \in L : (x * y) \leq_{\mathcal{L}} z$ iff $y \leq_{\mathcal{L}} (x \Rightarrow z)$, where $\leq_{\mathcal{L}}$ is the lattice usual order.

The second property tells us that given an associative and commutative function $* : L^2 \rightarrow L$, then $x * 1 = x$, for all $x \in L$.

In [GCDK07], it is seen that the following properties are valid, for all x, y, z in the residuated lattice L, where the notations $\neg x$ and $x \Leftrightarrow y$ mean, respectively, $x \Rightarrow 0$ and $(x \Rightarrow y) \wedge (y \Rightarrow x)$:

1. $x \Rightarrow y$ is equal to the largest element z in L that satisfies $x * z \leq_{\mathcal{L}} y$, so, we have $x \Rightarrow y = sup\{z \in L : x * z \leq_{\mathcal{L}} y\}$;
2. $x \leq_{\mathcal{L}} \neg\neg x$;
3. If $x \leq_{\mathcal{L}} y$, then $\neg y \leq_{\mathcal{L}} \neg x$;
4. $x \leq_{\mathcal{L}} y$ iff $x \Rightarrow y = 1$;
5. $x = y$ iff $x \Leftrightarrow y = 1$;
6. $x \leq_{\mathcal{L}} y \Rightarrow (x * y)$;
7. $x * (x \Rightarrow y) \leq_{\mathcal{L}} y$ (in particular: $x * \neg x = 0$);
8. $x * (y \vee z) = (x * y) \vee (x * z)$;

9. $\neg(x \vee y) = \neg x \wedge \neg y$;

10. $\neg(x * y) = x \Rightarrow \neg y$.

    Moreover, in residuated lattices with involutive negation ($\neg\neg x = x$), for every $x, y \in L$:

11. $\neg(x \wedge y) = \neg x \vee \neg y$;

12. $x \Rightarrow y = \neg y \Rightarrow \neg x$;

13. $\neg(x \Rightarrow y) = x * \neg y$.

Properties 9 and 11 are best known as the de Morgan laws .

The most well known residuated lattices are those on the unit interval $[0, 1]$, with the usual ordering. Besides, it is easily observed that $*$ is a t-norm on $[0, 1]$ if $\langle [0, 1], min, max, *, \Rightarrow, 0, 1\rangle$ is a residuated lattice [GCDK07].

Although in literature, e.g. [Höh95], the name residuated lattice is often used for more general structures, there is a different terminology, namely bounded integral commutative residuated lattice, or BIC-lattice for short, which refers to more specific structures (e.g. [GMO04] and [GCDK07]). We understand both terminologies refer to the same structure and, from now on, we will conveniently substitute the term BIC-lattice for residuated lattice.

Residuated lattices are called **complete** if the algebra $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ satisfies the following property:

iv. $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice.

A structure $\langle L, \wedge, \vee, ', 0, 1\rangle$ is named a **complemented** lattice if $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice and it also satisfies the condition below:

v. For all $x \in L$, $x \vee x' = 1$ and $x \wedge x' = 0$.

In this case, x' is called the complement of x.

Finally, a lattice is said to be **distributive** if both properties below are true:

vi. For all $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

vii. For all $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Some algebras can be defined using some of the classes shown above. Hence, a complemented and distributive lattice can define a **Boolean algebra**, which is a structure $\langle L, \wedge, \vee, ', 0, 1\rangle$ such that, for all $x, y, z \in L$:

1. $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;
2. $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$;
3. $x \vee x = x$ and $x \wedge x = x$;
4. $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$;
5. For all $x \subseteq L$, *sup* x exist;
6. For all $x \in L$, $x' \in L$ exist such that $x \vee x' = 1$ and $x \wedge x' = 0$;
7. For all $x, y, z \in L$, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;
8. For all $x, y, z \in L$, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

It can be seen that the four first properties indicate $\langle L, \wedge, \vee\rangle$ as a lattice. The fifth one indicates that $(L, \wedge, \vee, 0, 1)$ is complete. A complemented lattice $\langle L, \wedge, \vee, ', 0, 1\rangle$ is assured by the sixth condition. Finally, conditions 7 and 8 guarantee a distributive lattice.

At last, a **De Morgan algebra** is a distributive complemented lattice $\mathcal{L} = \langle L, \wedge, \vee, \neg, 0, 1\rangle$ such that the following properties are held:

- $\neg(x \vee y) = \neg x \wedge y$ and $\neg(x \wedge y) = \neg x \vee \neg y$;
- $\neg(\neg x) = x$.

3.2 A Semantic for Fuzzy Logic using the Lattice Theory

Since t-norms are a relevant issue in fuzzy logics, a good generalization of them is undoubtedly important. In [BS05], we obtained a lattice interpretation of the classic propositional connectives by generalizing some t-norms to bounded lattices. In this way:

Definition 3.1 *The quintuple $T_{\mathbb{L}} = \langle T, N, S, P, B\rangle$ is a fuzzy generalization of the classic propositional connectives to the bounded lattice framework, or $\mathbb{L}$-interpretation for short.*

Nevertheless, we concluded that given a lattice $\mathbb{L} = \langle L, \wedge, \vee, 0, 1\rangle$ and a fuzzy generalization $T_{\mathbb{L}}$:

i. T models the conjunction;

ii. N models the negation;

iii. S models the disjunction;

iv. P models the implication; and

v. B models the bi-implication.

Their definitions will be seen on the next subsections.

3.2.1 Lattice T-norm

In [BCBS07], we defined a lattice t-norm as follows:

Definition 3.2 *Let $\mathcal{L}$ be a bounded lattice. A binary operation T on L is a triangular norm on $\mathcal{L}$, t-norm in short, if for each $x, y, z \in L$ the following properties are satisfied:*

1. *Commutativity: $T(x, y) = T(y, x)$;*

2. *Associativity:* $T(x, T(y, z)) = T(T(x, y), z)$*;*

3. *Neutral element:* $T(x, 1) = x$*; and*

4. *Monotonicity: If* $y \leq_{\mathcal{L}} z,$ *then* $T(x, y) \leq_{\mathcal{L}} T(x, z)$*.*

The class of t-norms on a same bounded lattice can be partially ordered. Let T_1 and T_2 be t-norms on a bounded lattice $\mathcal{L}$. Then, T_1 is weaker than T_2 or, equivalently, T_2 is stronger than T_1, denoted by: $T_1 \leq_{\mathcal{L}} T_2$, if for each $x, y \in L, T_1(x, y) \leq_{\mathcal{L}} T_2(x, y)$.

3.2.2 Lattice Negation

In order to obtain a lattice negation, N, on a bounded lattice $\mathcal{L}$, we have:

Definition 3.3 *Let $\mathcal{L}$ be a bounded lattice. A unary operation N on L is a negation on $\mathcal{L}$ if, for each $x, y \in L$, the following properties are satisfied:*

1. $N(0) = 1$ *and* $N(1) = 0$*;*

2. *If* $x \geq_{\mathcal{L}} y$*, then* $N(x) \leq_{\mathcal{L}} N(y)$*.*

A negation can be called **strong lattice negation** if it also satisfies a third property: $N(N(x)) = x$, for all $x \in L$.

3.2.3 Lattice T-conorm

Analogously, we can define a triangular conorm on a bounded lattice $\mathcal{L}$.

Definition 3.4 *Let $\mathcal{L}$ be a bounded lattice. A binary operation S on L is a triangular conorm on $\mathcal{L}$, t-conorm in short, if for each $x, y, z \in L$, the following properties are satisfied:*

1. *Commutativity:* $S(x, y) = S(y, x)$*;*

2. *Associativity:* $S(x, S(y, z)) = S(S(x, y), z)$*;*

3. *Neutral element:* $S(x, 0) = x$*; and*

4. *Monotonicity: If* $y \leq_{\mathcal{L}} z$*, then* $S(x, y) \leq_{\mathcal{L}} S(x, z)$*.*

Proposition 3.3 *Let* $\mathcal{L}$ *be a bounded lattice,* T *a t-norm on* $\mathcal{L}$ *and* N *a strong negation on* $\mathcal{L}$*. Then,* $S_T(x, y) = N(T(N(x), N(y)))$ *is a t-conorm on* $\mathcal{L}$*.*

Proof: For all $x, y, z \in L$,

Commutativity:

$$\begin{aligned} S_T(x, y) &= N(T(N(x), N(y))), \text{ by the definition of } S_T. \\ &= N(T(N(y), N(x))), \text{ by the commutativity of } T. \\ &= S_T(y, x), \text{ by the definition of } S_T. \end{aligned}$$

Associativity:

$$\begin{aligned} S_T(x, S_T(y, z)) &= N(T(N(x), N(S_T(y, z)))), \text{ by the definition of } S_T. \\ &= N(T(N(x), N(N(T(N(y), N(z)))))), \text{ by the definition of } S_T. \\ &= N(T(N(x), T(N(y), N(z)))), \text{ by the strong negation.} \\ &= N(T(T(N(x), N(y)), N(z))), \text{ by the associativity of } T. \\ &= N(T(N(N(T((N(x), N(y)), N(z)))))), \text{ by the strong negation.} \\ &= S_T(N(T(N(x), N(y))), z), \text{ by the definition of } S_T. \\ &= S_T(S_T(x, y), z), \text{ by the definition of } S_T. \end{aligned}$$

Neutral element:

$$\begin{aligned} S_T(x, 0) &= N(T(N(x), N(0))) \\ &= N(T(N(x), 1)), \text{ by the negation definition.} \\ &= N(N(x)), \text{ by the neutral element property of } T. \\ &= x \end{aligned}$$

Monotonicity:

We know that $S_T(x,y) = N(T(N(x),N(y)))$ and $S_T(x,z) = N(T(N(x),N(z)))$, and by the monotonicity of T, such that $y \leq_{\mathcal{L}} z$, we can conclude that $N(T(N(x),N(y)))$ $\leq_{\mathcal{L}} N(T(N(x),N(z)))$, that is $S_T(x,y) \leq_{\mathcal{L}} S_T(x,z)$.

Or, in a different way, we can see that: $y \leq z \Rightarrow N(y) \geq N(z)$;
$T(N(x),N(y)) \geq N(T(N(x),N(z))$;
$N(T(N(x),N(y))) \leq N(T(N(x),N(z)))$;
$S_T(x,y) \leq S_T(x,z)$. ■

3.2.4 Lattice Implication

A lattice implication can also be defined in a bounded lattice. As there are a lot of definitions for fuzzy implication, together with the related properties, we emphasize the ones listed below.

Definition 3.5 *Let $\mathcal{L}$ be a bounded lattice. A binary operation P on L is an implication on $\mathcal{L}$ if, for each $x, y, z \in L$:*

1. *$x \leq_{\mathcal{L}} z$ implies $P(x,y) \geq_{\mathcal{L}} P(z,y)$, for all $x, y, z \in L$;*

2. *$y \leq_{\mathcal{L}} z$ implies $P(x,y) \leq_{\mathcal{L}} P(x,z)$, for all $x, y, z \in L$;*

3. *$P(0,y) = 1$, for all $y \in L$;*

4. *$P(x,1) = 1$, for all $y \in L$;*

5. *$P(1,0) = 0$.*

In [GCDK07, prop.2] (proof in [Tur99]), it was stated that:

Proposition 3.4 *Let $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. Then the following property is valid, for every x, y and $z \in L$:*

(1) $x \Rightarrow y = sup\{z \in L : x * z \leq_{\mathcal{L}} y\}$.

This proposition shows that the $\Rightarrow$ operation (or P) in a residuated lattice is completely determined from the $*$ operation (or T).

Proposition 3.5 *Let $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. Then the following property is valid, for every x, y and $z \in L$:*

(1) $x * y = inf\{z \in L : x \leq_{\mathcal{L}} (y \Rightarrow z)\}$.

Proof: This follows from the fact that the relation between $*$ and $\Rightarrow$ is based on Galois connection [GHK+80]. ■

Proposition 3.6 *Let $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. Then $*$ is a t-norm on L.*

Proof: Once $\langle L, *, 1\rangle$ is a monoid, then it only remains to prove that it is monotonic. Let $y \leq_{\mathcal{L}} y'$, then, by the first property of definition 3.5, we have $y \Rightarrow z \geq_{\mathcal{L}} y' \Rightarrow z$. So, $\{z \in L : x \leq_{\mathcal{L}} y \Rightarrow z\} \supseteq \{z \in L : x \leq_{\mathcal{L}} y' \Rightarrow z\}$. Therefore, $inf\{z \in L : x \leq_{\mathcal{L}} y \Rightarrow z\} \leq_{\mathcal{L}} inf\{z \in L : x \leq_{\mathcal{L}} y' \Rightarrow z\}$, and by proposition 3.5, $x * y \leq_{\mathcal{L}} x * y'$. ■

Note that the residuum of a t-norm $*$ is indeed an implication, thus it satisfies the properties on definition 3.5.

Proposition 3.7 *Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a bounded lattice and $*$ a t-norm on L. Then, $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow_*, 0, 1\rangle$ is a residuated lattice.*

Proof: We know $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow_*, 0, 1\rangle$ is bounded and, from definition 3.2, it is clear the associativity and the neutral element guarantee that $\langle L, *, 1\rangle$ is a commutative monoid.

According to the definition of residuated lattice, it only remains to prove that $(*, \Rightarrow_*)$ form an adjoint pair. So, $x * y \leq z$ iff $y \in \{u \in L : x * u \leq_{\mathcal{L}} z\}$ iff $y \leq_{\mathcal{L}} sup\{u \in L : x * u \leq_{\mathcal{L}} z\}$ iff $y \leq_{\mathcal{L}} x \Rightarrow_* z$. ■

Corollary 3.1 *Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a bounded lattice. Then $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a residuated lattice iff $*$ is a t-norm on L and $\Rightarrow = \Rightarrow_*$.*

Proof: Straightforward from propositions 3.4, 3.6 and 3.7. ■

In case $\Rightarrow$ is a lattice R-implication based on a left-continuous t-norm on $\mathcal{L}$, then properties i. to iv. hold for $\Rightarrow$, where iii. and iv. are seen below:

iii. $x \Rightarrow (y \Rightarrow z) = y \Rightarrow (x \Rightarrow z)$, $x, y, z \in L$ (exchange principle);

iv. $x \leq_{\mathcal{L}} y$ iff $x \Rightarrow y = 1$, $x, y \in L$ (ordering property).

3.2.5 Lattice Bi-Implication

The lattice bi-implication, in the bounded lattice $\mathcal{L}$, can be defined as well:

Definition 3.6 *Let $\mathcal{L}$ be a bounded lattice. A binary operation B on L is a bi-implication on $\mathcal{L}$ if, for each $x, y, z \in L$:*

1. $B(x, y) = B(y, x)$;
2. *If* $x = y$, *then* $B(x, y) = 1$;
3. $B(0, 1) = 0$;
4. *If* $x \leq_{\mathcal{L}} y \leq_{\mathcal{L}} z$, *then* $B(x, y) \geq_{\mathcal{L}} B(x, z)$ *and* $B(y, z) \geq_{\mathcal{L}} B(x, z)$.

3.3 The Interval Constructor on Lattices

The pursue of dealing with uncertainty, aiming the control of errors in a computation, has contributed to the growing interest on interval mathematics, IM for short, whose main goal is to give a mathematical foundation to interval computations. It was firstly introduced by R. E. Moore in [Moo59] and, in this context, real values (completely known or not) are represented by closed intervals (containing their values), instead of being represented by floating-point numbers. IM can guarantee, through the maximum exactness principle (directed rounding) and the optimal scalar product, that the resulting interval, obtained after the process of interval computation, contains the punctual result - meaning it is an approximation of the exact result - [Kul99],[Hay03], [BDR04], where the diameter of the interval is an indication of the maximum error that could have been occurred. In this way, the interval computation permits an automatic and rigid control of errors in numerical computation. For instance, errors of initial data modeling, as well as rounding and errors of the computational process [Moo59], [Moo79],[Kul99],[Hay03]. Besides, IM has been extensively used to represent uncertain data or qualitative values in different areas, such as Scientific and Technological Computation, Artificial Intelligence, Soft Computing, etc. ([KK96],[KNDB03],[ADC04], [BDR04], [DC04]). Many applications of the IM can be found at the address: http://www.cs.utep.edu/interval-comp/, or in the work of Kearfott and Kreinovich [KK96].

The compatibility of IM and fuzzy logics seems reasonable, once the uncertainty can also be in the membership degree. If an expert, for example, is uncertain about something, he can also be uncertain about the degree of belief. Hence, we can understand the relevance of the interval-valued fuzzy set theory, introduced by [Jah75], [Zad75], among other authors, where the traditional $[0, 1]$-valued membership degrees were replaced by intervals in $[0, 1]$.

3.3.1 Interval Constructor

It is not difficult to see that an interval has a dual nature. It can be seen as a set of real numbers, but also, it can be understood as an ordered pair of real numbers.

Let $\mathbb{IR}$ be the set of intervals with real numbers as end points, where $\mathbb{IR} = \{[x, y] : x, y \in \mathbb{R} \text{ and } x \leq y\}$. $\mathbb{IR}$ is associated with two projections: $\pi_a : \mathbb{IR} \rightarrow \mathbb{R}$ and $\pi_b : \mathbb{IR} \rightarrow \mathbb{R}$. The projections π_a and π_b are defined as follows:

$$\pi_a([a, b]) = a \text{ and } \pi_b([a, b]) = b$$

For any interval variable X, $\pi_a(X)$ and $\pi_b(X)$ will be denoted, as a convention, by $\underline{x}$ and $\overline{x}$, respectively. Thus, $X = [\underline{x}, \overline{x}]$.

Despite real numbers have a natural total order, the usual order on $\mathbb{IR}$ are all non-total orders. These partial orders can be defined taking into account the different natures of intervals, for instance, if it is seen as a set of $\mathbb{R}$, the natural partial order is the inclusion (introduced in [Sun58]). Then, formally, for each $X, Y \in \mathbb{IR}$, $X \subseteq Y \Leftrightarrow \underline{y} \leq \underline{x} \leq \overline{x} \leq \overline{y}$. When an interval is considered as ordered pair of real numbers, then the natural order is the one inherited from the Cartesian product order, first considered by Kulisch and Miranker in [KM81]. Formally, for each $X, Y \in \mathbb{IR}$: $X \leq Y \Leftrightarrow \underline{x} \leq \underline{y}$ and $\overline{x} \leq \overline{y}$. Finally, if an interval is dealt as a representation of an unknown real number, or simply information, the natural order is the one introduced by Scott in [Sco70], and also used by Acióly in [Aci91] to provide interval mathematics with a computational foundation. So, for each $X, Y \in \mathbb{IR}$: $X \sqsubseteq Y \Leftrightarrow \underline{x} \leq \underline{y} \leq \overline{y} \leq \overline{x}$.

Observe that the real intervals and also the order on the set of real intervals depend on the usual order in $\mathbb{R}$. In [CBB00], the interval constructor was formalized (considering any partially ordered set) on the category **POSet**, as follows:

Definition 3.7 *Let $\mathcal{L} = \langle L, \leq \rangle$ be a poset. The poset $\mathbb{I}(\mathcal{L}) = \langle \mathbb{I}[L], \leq_{\mathbb{I}} \rangle$, where*

- $\mathbb{I}[L] = \{X : \underline{x}, \overline{x} \in L \text{ and } \underline{x} \leq \overline{x}\}$
- $X \leq_{\mathbb{I}} Y \Leftrightarrow \underline{x} \leq \underline{y} \text{ and } \overline{x} \leq \overline{y}$

is called the ***poset of intervals of*** $\mathcal{L}$*.*

There are also two natural functions from $\mathbb{I}[L]$ to $L,$ the left and right projections π_a : $\mathbb{I}[L] \rightarrow L$ and $\pi_b : \mathbb{I}[L] \rightarrow L$, respectively. They were also defined by: $\pi_a(X) = \underline{x}$ and $\pi_b(X) = \overline{x}$.

It is clear that, despite of their dual nature, the intervals have quite similar definitions. And we will use the second point of view, as we are mostly dealing with Kulisch and Miranker order. So, from now on, our interval constructor will be represented by $\mathbb{I}$.

3.3.2 Interval Constructor on Classes of Lattices

In order to obtain the desired idea of an **interval residuated lattice**, $\mathbb{I}[L]$ for short, we need definition 3.7, where $\mathbb{I}[L]$ is our interval constructor $\mathbb{I}$ applied to a residuated lattice L. Hence, we firstly must introduce our interval constructor on the classes of lattices seen on the previous chapter.

Proposition 3.8 *Let* $\mathcal{L} = \langle L, \wedge, \vee\rangle$ *be a lattice. Then,* $\mathbb{I}[\mathcal{L}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}\rangle$*, where:*

- $\mathbb{I}[L] = \{X : \underline{x}, \overline{x} \in L \text{ and } \underline{x} \leq_{\mathcal{L}} \overline{x}\}$,
- $X \sqcap_{\mathbb{I}} Y = [\underline{x} \wedge \underline{y}, \overline{x} \wedge \overline{y}]$ *and*
- $X \sqcup_{\mathbb{I}} Y = [\underline{x} \vee \underline{y}, \overline{x} \vee \overline{y}]$,

is also a lattice.

Proof:

Commutativity:

$$\begin{aligned}
X \sqcap_{\mathbb{I}} Y &= Y \sqcap_{\mathbb{I}} X \\
&= [\underline{y}, \overline{y}] \sqcap_{\mathbb{I}} [\underline{x}, \overline{x}], \text{ by proposition 3.8.} \\
&= [\underline{y} \wedge \underline{x}, \overline{y} \wedge \overline{x}], \text{ by proposition 3.8.} \\
&= [\underline{x} \wedge \underline{y}, \overline{x} \wedge \overline{y}], \text{ by the commutativity of L.} \\
&= [\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} [\underline{y}, \overline{y}], \text{ by proposition 3.8.} \\
&= X \sqcap_{\mathbb{I}} Y, \text{ by proposition 3.8.}
\end{aligned}$$

and

$$\begin{aligned}
X \sqcup_{\mathbb{I}} Y &= Y \sqcup_{\mathbb{I}} X \\
&= [\underline{y}, \overline{y}] \sqcup_{\mathbb{I}} [\underline{x}, \overline{x}], \text{ by proposition 3.8.} \\
&= [\underline{y} \vee \underline{x}, \overline{y} \vee \overline{x}], \text{ by proposition 3.8.} \\
&= [\underline{x} \vee \underline{y}, \overline{x} \vee \overline{y}], \text{ by the commutativity of L.} \\
&= [\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} [\underline{y}, \overline{y}], \text{ by proposition 3.8.} \\
&= X \sqcup_{\mathbb{I}} Y;
\end{aligned}$$

Associativity:

$$\begin{aligned}
X \sqcap_{\mathbb{I}} (Y \sqcap_{\mathbb{I}} Z) &= (X \sqcap_{\mathbb{I}} Y) \sqcap_{\mathbb{I}} Z \\
&= [\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} ([\underline{y}, \overline{y}] \sqcap_{\mathbb{I}} [\underline{z}, \overline{z}]) \\
&= [\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} [\underline{y} \wedge \underline{z}, \overline{y} \wedge \overline{z}] \\
&= [\underline{x} \wedge (\underline{y} \wedge \underline{z}), \overline{x} \wedge (\overline{y} \wedge \overline{z})] \\
&= [(\underline{x} \wedge \underline{y}) \wedge \underline{z}, (\overline{x} \wedge \overline{y}) \wedge \overline{z}], \text{by the associativity of L.} \\
&= ([\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} [\underline{y}, \overline{y}]) \sqcap_{\mathbb{I}} [\underline{z}, \overline{z}] \\
&= (X \sqcap_{\mathbb{I}} Y) \sqcap_{\mathbb{I}} Z
\end{aligned}$$

and

$$\begin{aligned}
X \sqcup_{\mathbb{I}} (Y \sqcup_{\mathbb{I}} Z) &= (X \sqcup_{\mathbb{I}} Y) \sqcup_{\mathbb{I}} Z \\
&= [\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} ([\underline{y}, \overline{y}] \sqcup_{\mathbb{I}} [\underline{z}, \overline{z}]) \\
&= [\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} [\underline{y} \vee \underline{z}, \overline{y} \vee \overline{z}] \\
&= [\underline{x} \vee (\underline{y} \vee \underline{z}), \overline{x} \vee (\overline{y} \vee \overline{z})] \\
&= [(\underline{x} \vee \underline{y}) \vee \underline{z}, (\overline{x} \vee \overline{y}) \vee \overline{z}], \text{ by the associativity of L.} \\
&= ([\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} [\underline{y}, \overline{y}]) \sqcup_{\mathbb{I}} [\underline{z}, \overline{z}] \\
&= (X \sqcup_{\mathbb{I}} Y) \sqcup_{\mathbb{I}} Z;
\end{aligned}$$

Idempotency:

$$\begin{aligned}
X \sqcap_{\mathbb{I}} X &= X \\
&= [\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} ([\underline{x}, \overline{x}] \\
&= [\underline{x} \wedge \underline{x}, \overline{x} \wedge \overline{x}] \\
&= [\underline{x}, \overline{x}], \text{ by the idempotency of L.} \\
&= X
\end{aligned}$$

and

$$\begin{aligned}
X \sqcup_{\mathbb{I}} X &= X \\
&= [\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} ([\underline{x}, \overline{x}] \\
&= [\underline{x} \vee \underline{x}, \overline{x} \vee \overline{x}] \\
&= [\underline{x}, \overline{x}], \text{ by the idempotency of L.} \\
&= X;
\end{aligned}$$

Absorption:

$$\begin{aligned}
X \sqcap_{\mathbb{I}} (X \sqcup_{\mathbb{I}} Y) &= X \\
&= [\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} ([\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} [\underline{y}, \overline{y}]) \\
&= [\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} [\underline{x} \vee \underline{y}, \overline{x} \vee \overline{y}] \\
&= [\underline{x} \wedge (\underline{x} \vee \underline{y}), \overline{x} \wedge (\overline{x} \vee \overline{y})] \\
&= [(\underline{x}, \overline{x}], \text{ by the absorption of L.} \\
&= X
\end{aligned}$$

and

$$
\begin{aligned}
X \sqcup_{\mathbb{I}} (X \sqcap_{\mathbb{I}} Y) &= X \\
&= [\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} ([\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} [\underline{y}, \overline{y}]) \\
&= [\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} [\underline{x} \wedge \underline{y}, \overline{x} \wedge \overline{y}] \\
&= [\underline{x} \vee (\underline{x} \wedge \underline{y}), \overline{x} \vee (\overline{x} \wedge \overline{y})] \\
&= [(\underline{x}, \overline{x}], \text{ by the absorption of L.} \\
&= X. \blacksquare
\end{aligned}
$$

Remark 3.1 *The order of the lattice $\mathbb{I}[\mathcal{L}]$ coincides with the interval constructor order on the lattice $\langle L, \leq\rangle$, that is: $\leq_{\mathbb{I}[\mathcal{L}]} \; = \; \leq_{\mathbb{I}}$.*

Proposition 3.9 *Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a bounded lattice. Then, $\mathbb{I}[\mathcal{L}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, [0,0], [1,1]\rangle$ is also a bounded lattice, where: $[1,1]$ is the top of $\mathcal{L}$ and $[0,0]$ is the bottom of $\mathcal{L}$.*

Proof: Since, by proposition 3.8, $\mathbb{I}[\mathcal{L}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}\rangle$ is a lattice, then we only must prove that $\mathbb{I}[\mathcal{L}]$ also satisfies the following two properties:

- For all $X \in \mathbb{I}[L]$, $X \sqcap_{\mathbb{I}} [1,1] = X$ and $X \sqcup_{\mathbb{I}} [1,1] = [1,1]$;
- For all $X \in \mathbb{I}[L], X \sqcap_{\mathbb{I}} [0,0] = [0,0]$ and $X \sqcup_{\mathbb{I}} [0,0] = X$.

Thus, let $X \in \mathbb{I}[L]$, then:

$$
\begin{aligned}
X \sqcap_{\mathbb{I}} [1,1] &= [\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} [1,1] \\
&= [\underline{x} \wedge 1, \overline{x} \wedge 1] \\
&= [\underline{x}, \overline{x}] \\
&= X;
\end{aligned}
$$

$$
\begin{aligned}
X \sqcup_{\mathbb{I}} [1,1] &= [\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} [1,1] \\
&= [\underline{x} \vee 1, \overline{x} \vee 1] \\
&= [1,1];
\end{aligned}
$$

$$\begin{aligned} X \sqcap_{\mathbb{I}} [0,0] &= [\underline{x}, \overline{x}] \sqcap_{\mathbb{I}} [0,0] \\ &= [\underline{x} \wedge 0, \overline{x} \wedge 0] \\ &= [0,0] \end{aligned}$$

and

$$\begin{aligned} X \sqcup_{\mathbb{I}} [0,0] &= [\underline{x}, \overline{x}] \sqcup_{\mathbb{I}} [0,0] \\ &= [\underline{x} \vee 0, \overline{x} \vee 0] \\ &= [\underline{x}, \overline{x}] \\ &= X. \blacksquare \end{aligned}$$

Lemma 3.1 *Let $\langle L, *, 1\rangle$ be a commutative monoid on a poset $\mathcal{L}$, then $\langle \mathbb{I}[L], \mathbb{I}[*], [1,1]\rangle$ is a commutative monoid on the poset $\mathbb{I}[\mathcal{L}]$, where the operation $\mathbb{I}[*]$ is defined as follows: $X \; \mathbb{I}[*] \; Y = [\underline{x} * \underline{y}, \overline{x} * \overline{y}]$.*

Proof: Straightforward from definition of $\mathbb{I}[*]$ and the fact that $\langle L, *, 1\rangle$ is a commutative monoid, we know $\mathbb{I}[*]$ is commutative and it has $[1,1]$ as neutral element. However, the associativity of $\mathbb{I}[*]$ deserves a bit of attention:

$$\begin{aligned} X \; \mathbb{I}[*] \; (Y \; \mathbb{I}[*] \; Z) &= X \; \mathbb{I}[*] \; [\underline{y} * \underline{z}, \overline{y} * \overline{z}] \\ &= [\underline{x} * (\underline{y} * \underline{z}), \overline{x} * (\overline{y} * \overline{z})] \\ &= [(\underline{x} * \underline{y}) * \underline{z}, (\overline{x} * \overline{y}) * \overline{z}] \\ &= ([\underline{x} * \underline{y}, \overline{x} * \overline{y}]) \; \mathbb{I}[*] \; Z \\ &= (X \; \mathbb{I}[*] \; Y) \; \mathbb{I}[*] \; Z. \blacksquare \end{aligned}$$

Proposition 3.10 *Let $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. Then, $\mathbb{I}[\mathcal{L}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ is also a residuated lattice, where $X \; \mathbb{I}[\Rightarrow] \; Y = [\overline{x} \Rightarrow \underline{y}, \underline{x} \Rightarrow \overline{y}]$. Moreover, $\mathbb{I}[*]$ and $\mathbb{I}[\Rightarrow]$ form an adjoint pair.*

Proof: By proposition 3.9, $\mathbb{I}[\mathcal{L}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, [0,0], [1,1]\rangle$ is a bounded lattice and by lemma 3.1, $\langle \mathbb{I}[L], \mathbb{I}[*], [1,1]\rangle$ is a commutative monoid. Then, it would only remain to prove that $\mathbb{I}[\Rightarrow]$ is a function $\mathbb{I}[L]^2 \rightarrow \mathbb{I}[L]$ such that, for all $X, Y, Z \in \mathbb{I}[L] : (X \; \mathbb{I}[*] \; Y) \leq_{\mathbb{I}} Z$ iff $Y \leq_{\mathbb{I}} (X \; \mathbb{I}[\Rightarrow] \; Z)$. However, this proof is analogous to the one used to prove proposition 3.7, as $\mathcal{L}$ is a residuated lattice. $\blacksquare$

3.3.3 The Interval Constructor on Fuzzy Connectives

In [BCBS07], we obtained an important result showing how to transform an arbitrary t-norm, on a bounded lattice, into a t-norm on its interval bounded lattice.

Proposition 3.11 *Let $*$ be a t-norm on the bounded lattice L. Then $\mathbb{I}[*] : \mathbb{I}[L]^2 \to \mathbb{I}[L]$ defined by*

$$X \ \mathbb{I}[*] \ Y = [\underline{x} * \underline{y}, \overline{x} * \overline{y}]$$

is a t-norm on the bounded lattice $\mathbb{I}[L]$.

Proof: It is easy to see that the commutativity, monotonicity and neutral element $([1,1])$ properties of $\mathbb{I}[*]$ follow straightforward from the same properties of the t-norm $*$. Then, it only remains to prove the associativity property, which is shown below:

$$\begin{aligned}
X \ \mathbb{I}[*](Y \ \mathbb{I}[*] \ Z) &= X \ \mathbb{I}[*] \ [\underline{y} * \underline{z}, \overline{y} * \overline{z}] \\
&= [\underline{x} * (\underline{y} * \underline{z}), \overline{x} * (\overline{y} * \overline{z})] \\
&= [(\underline{x} * \underline{y}) * \underline{z}, (\overline{x} * \overline{y}) * \overline{z}] \\
&= ([\underline{x} * \underline{y}, \overline{x} * \overline{y}]) \ \mathbb{I}[*] \ Z \\
&= (X \ \mathbb{I}[*] \ Y) \ \mathbb{I}[*] \ Z. \ \blacksquare
\end{aligned}$$

So, if $*$ is a t-norm on the bounded lattice L, then $\mathbb{I}[*]$ is an interval t-norm on the bounded lattice $\mathbb{I}[L]$.

Using the same idea, we are going to introduce here some other definitions, in order to obtain an interval t-conorm on a bounded lattice, as well as the interval implication and negation. Thus, to define how to transform an arbitrary t-conorm on a bounded lattice into a t-conorm on its interval bounded lattice, we have the following proposition:

Proposition 3.12 *Let S be a t-conorm on the bounded lattice L. Then $\mathbb{I}[S] : \mathbb{I}[L]^2 \to \mathbb{I}[L]$ defined by*

$$\mathbb{I}[S](X,Y) = [S(\underline{x},\underline{y}), S(\overline{x},\overline{y})]$$

is a t-conorm on bounded lattice $\mathbb{I}[L]$.

Proof: It is also easy to see that the commutativity, monotonicity and neutral element $([0,0])$ properties of $\mathbb{I}[S]$ follow straightforward from the same properties of the t-conorm S. We must observe though, the associativity property, which is shown below:

$$\begin{aligned}
\mathbb{I}[S](X,\mathbb{I}[S](Y,Z)) &= \mathbb{I}[S](X,[S(\underline{y},\underline{z}),S(\overline{y},\overline{z})]) \\
&= [S(\underline{x},S(\underline{y},\underline{z})),S(\overline{x},S(\overline{y},\overline{z}))] \\
&= [S(S(\underline{x},\underline{y}),\underline{z}),S(S(\overline{x},\overline{y}),\overline{z})] \\
&= \mathbb{I}[S]([S(\underline{x},\underline{y}),S(\overline{x},\overline{y})],Z) \\
&= \mathbb{I}[S](\mathbb{I}[S](X,Y),Z). \blacksquare
\end{aligned}$$

So, if S, or analogously $\oplus$ (infix notation), is a t-conorm on the bounded lattice $\mathcal{L}$, then $\mathbb{I}[S]$, or $\mathbb{I}[\oplus]$, is an interval t-conorm on the bounded lattice $\mathbb{I}[L]$.

An arbitrary implication $\Rightarrow$, on a bounded lattice, can be transformed as well, into an implication on its interval bounded lattice, represented here by $\mathbb{I}[\Rightarrow]$. Despite of the many different definitions for fuzzy implications and their related properties, we will only consider some of them. The following proposition shows us how to achieve $\mathbb{I}[\Rightarrow]$, based on the work of Bedregal et. al. ([BSDR07]):

Proposition 3.13 *A function* $\mathbb{I}[\Rightarrow] : \mathbb{I}[L]^2 \rightarrow \mathbb{I}[L]$ *is an interval fuzzy implication if the following conditions are satisfied, for all* $X, Y, Z \in \mathbb{I}[L]$*:*

1. $X \leq_{\mathbb{I}} Z$ *implies* $X \ \mathbb{I}[\Rightarrow] \ Y \geq_{\mathbb{I}} Z \ \mathbb{I}[\Rightarrow] \ Y$*;*

2. $Y \leq_{\mathbb{I}} Z$ *implies* $X \ \mathbb{I}[\Rightarrow] \ Y \leq_{\mathbb{I}} X \ \mathbb{I}[\Rightarrow] \ Z$*;*

3. $[0,0] \ \mathbb{I}[\Rightarrow] \ Y = [1,1]$*;*

4. $X \ \mathbb{I}[\Rightarrow] \ [1,1] = [1,1]$*;*

5. $[1,1]\ \mathbb{I}[\Rightarrow]\ [0,0] = [0,0]$.

Proof: See [BSDR07]. ■

Another implication on the bounded lattice $\mathbb{I}[L]$, proposed in [BSDR07], can be seen below:

Proposition 3.14 *Let $\mathbb{I}[\Rightarrow]$ be an implication on the bounded lattice $\mathbb{I}[L]$. Then $\mathbb{I}[\Rightarrow]$, defined by*

$$X\ \mathbb{I}[\Rightarrow]\ Y = [\overline{x} \Rightarrow \underline{y}, \underline{x} \Rightarrow \overline{y}]$$

is an implication on the bounded lattice $\mathbb{I}[L]$ such that the following properties are satisfied:

1. $Y \leq_{\mathbb{I}} Z$ *implies* $X\ \mathbb{I}[\Rightarrow]\ Y \leq_{\mathbb{I}} X\ \mathbb{I}[\Rightarrow]\ Z$*;*
2. $X\ \mathbb{I}[\Rightarrow]\ (Y\ \mathbb{I}[\Rightarrow]\ Z) = Y\ \mathbb{I}[\Rightarrow]\ (X\ \mathbb{I}[\Rightarrow]\ Z)$*;*
3. $X\ \mathbb{I}[\Rightarrow]\ Y = [1,1]$*, iff* $\overline{x} \leq_{\mathbb{I}} \underline{y}$.

Proof: See [BSDR07]. ■

Proposition 3.15 *Let $*$ be a t-norm on a bounded lattice $\mathcal{L}$, then $(X \Rightarrow_{\mathbb{I}[*]} Y = sup\{Z \in \mathbb{I}[\mathcal{L}] : X\ \mathbb{I}[*]\ Z \leq_{\mathbb{I}} Y\}$ is a R-implication on $\mathbb{I}[\mathcal{L}]$. Moreover, $\Rightarrow_{\mathbb{I}[*]} = \mathbb{I}[\Rightarrow_*]$*

Proof: Analogous to [BSDR07, Theorem 6.9]. ■

Let $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. It is clear, by the corollary 3.1, that $(*, \Rightarrow)$ form an adjoint pair [EG01]. Then, to obtain the adjoint pair, after the addition of the interval constructor, is straightforward: $(\mathbb{I}[*], \mathbb{I}[\Rightarrow])$.

$$
\begin{array}{ccc}
* & \longrightarrow & \Rightarrow \\
\downarrow & & \downarrow \\
\mathbb{I}[*] & \longrightarrow & \mathbb{I}[\Rightarrow] = \Rightarrow_{\mathbb{I}[*]}
\end{array}
$$

Figure 3.1: Commutative diagram relating $*, \Rightarrow, \mathbb{I}[*]$ and $\mathbb{I}[\Rightarrow]$.

With these results, one important conclusion, obtained in [BSDR07], was the fact that the diagram shown in figure 3.1 above commutes. This diagram represents the idea of existing two possible ways to obtain $\mathbb{I}[\Rightarrow]$.

On one hand, we can add the interval constructor directly to $\Rightarrow$. On the other hand, we can first add it to the t-norm $*$ and then, from $\mathbb{I}[*]$, we can obtain $\mathbb{I}[\Rightarrow]$. It was proved $\mathbb{I}[\Rightarrow]$ and $\Rightarrow_{\mathbb{I}[*]}$ are equal.

Finally, transforming an arbitrary negation in a bounded lattice into a negation on its interval bounded lattice is done as follows.

Proposition 3.16 *Let N be a negation on the bounded lattice L. Then $\mathbb{I}[N] : \mathbb{I}[L] \rightarrow \mathbb{I}[L]$ defined by*

$$\mathbb{I}[N](X) = [N(\overline{x}), N(\underline{x})]$$

is a negation on the bounded lattice $\mathbb{I}[L]$.

Proof: The properties below are held:

1. Trivially, $\mathbb{I}[N]([0,0]) = [1,1]$ and $\mathbb{I}[N]([1,1]) = [0,0]$.

2. If $X \geq_{\mathbb{I}} Y$, then $\underline{x} \geq \underline{y}$ and $\overline{x} \geq \overline{y}$. So, by the antitonicity of N, $N(\underline{y}) \geq N(\underline{x}), N(\overline{y}) \geq N(\overline{x}), N(\underline{y}) \geq N(\overline{y})$ and $N(\underline{x}) \geq N(\overline{x})$. Hence, $[N(\overline{y}), N(\underline{y})] \geq_{\mathbb{I}} [N(\underline{x}), N(\overline{x})]$, i.e $\mathbb{I}[N](Y) \geq_{\mathbb{I}} \mathbb{I}[N](X)$. ■

Proposition 3.17 *If N is a strong lattice negation, then $\mathbb{I}[N]$ is also a strong negation.*

Proof: From the definition of lattice negation on the bounded lattice $\mathbb{I}[L]$, we have:

$$\begin{aligned}\mathbb{I}[N](\mathbb{I}[N](X)) &= \mathbb{I}[N](N(\overline{x}), N(\underline{x}))\\ &= [N(N(\underline{x})), N(N(\overline{x}))]\\ &= [\underline{x}, \overline{x}]\\ &= X, \text{ for all } X \in \mathbb{I}[L]. \blacksquare\end{aligned}$$

Most of the definitions and propositions given on this chapter, especially the ones where the interval constructor $\mathbb{I}$ was applied to classes of residuated lattices and fuzzy connectives, will be fundamental to prove that some classes of ML-algebras are preserved by the interval constructor. Monoidal Logic and its algebraic counterpart, namely ML-algebra, will be introduced on the next chapter.

Chapter 4

ML logic and ML-algebras

The following sections will discuss a little bit of Monoidal Logic (ML) and ML-algebra.

4.1 Monoidal Logic

Monoidal Logic, ML for short, introduced by Höhle in [Höh95], whose algebraic counterpart is the class of residuated lattices, gave a common framework to several first order non-classical logics, such as Linear logic, Łukasiewicz logic, among others. This logic was built up from the following primitive connectives: $*, \rightarrow, \wedge$ and $\vee$, and the truth constant $\overline{0}$ [1]. We can note that the connectives $\wedge$ and $\vee$ can not be defined from the others, and this is the reason why they were introduced as primitive connectives.

The axioms of ML are listed below:

- ML_1 $((\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma)))$, Syllogism Law;
- ML_2 $(\alpha \rightarrow (\alpha \vee \beta))$;
- ML_3 $(\beta \rightarrow (\alpha \vee \beta))$;

[1] Höhle uses negation as primitive connective, in the original formal system, in [Höh95]. However, we prefer to use the axiomatic system, given in [Got00], which is equivalent.

- ML_4 $((\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma)))$;
- ML_5 $((\alpha \wedge \beta) \to \alpha)$;
- ML_{5a} $((\alpha * \beta) \to \alpha)$;
- ML_6 $((\alpha \wedge \beta) \to \beta)$;
- ML_{6a} $((\alpha * \beta) \to (\beta * \alpha))$;
- ML_{6b} $(((\alpha * \beta) * \gamma) \to (\alpha * (\beta * \gamma)))$;
- ML_7 $((\gamma \to \alpha) \to ((\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta))))$;
- ML_8 $((\alpha \to (\beta \to \gamma)) \to ((\alpha * \beta) \to \gamma))$, Importation Law;
- ML_9 $(((\alpha * \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma)))$, Exportation Law;
- ML_{10} $((\alpha * \neg\alpha) \to \beta)$, Duns Scotus;
- ML_{11} $((\alpha \to (\alpha * \neg\alpha)) \to \neg\alpha)$.

The inference rule of ML is modus ponens. ML_1 is also known as the Syllogism Law, ML_8 and ML_9 are the Importation and Exportation Laws, respectively, and ML_{10} is known as Duns Scotus.

Some extensions of this logic seem quite interesting because there is a sort of interaction between these extensions and other logics. ML extensions include: MTL (Monoidal T-norm based Logic), introduced by Esteva and Godo in [EG01] (which is intended to cope with the tautologies of left-continuous t-norms and their residua); IMTL (Involutive Monoidal t-norm logic), WNM (Weak Nilpotent Minimum logic) and NM (Nilpotent Minimum logic), which are extensions of MTL.

In figure 4.1, there is a diagram of logics (and their axioms) and the relationships between ML and MTL with its extensions, BL (Basic Fuzzy logic, that is the many-valued residuated logic, introduced by Hájek in [Háj98]), Łukasiewicz logic, and Affine Multiplicative Linear logic, AMALL for short, a propositional fragment of Girard's Linear logic, in [Gir87] and [Tro92]. The arrows indicate the extensions labeled with the axioms added.

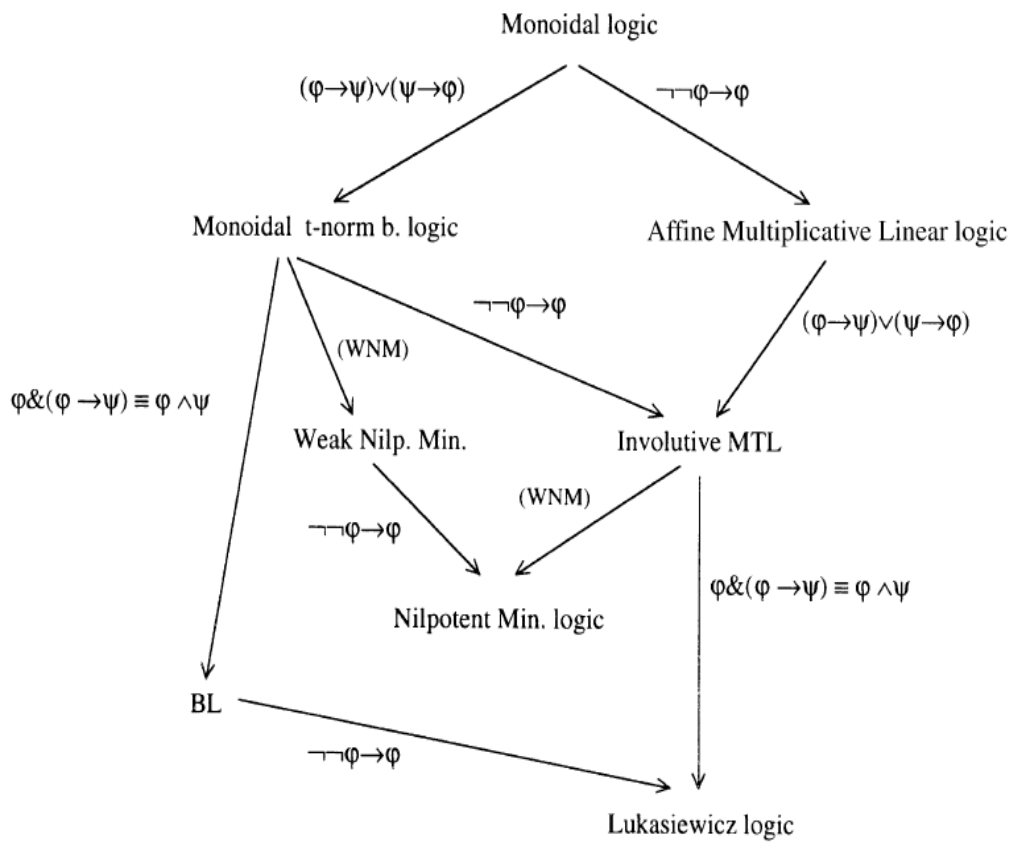

Figure 4.1: Diagram of logics and their relating axioms

Observing the diagram more carefully, we can notice that by adding the prelinearity axiom (PRL)$(\varphi \to \psi) \vee (\psi \to \varphi)$ and axiom (INV), defined by: (INV)$\neg\neg\varphi \to \varphi$, to Monoidal logic, we obtain MTL and AMALL, respectively. Analogously, IMTL was obtained either after axiom (PRL) was added to AMALL or axiom (INV) was added to MTL. Weak Nilpotent Minimum logic resulted by the addition of axiom (WNM) to MTL, where (WNM)$((\varphi \& \psi) \to 0) \vee ((\varphi \wedge \psi) \to (\varphi \& \psi))$. Nilpotent Minimum logic (NM) can be seen as an extension of WNM logic with axiom (INV) or an extension of IMTL logic with axiom (WNM). Finally, BL was obtained by adding the divisibility axiom (DIV) to MTL, where (DIV) $\varphi \& (\varphi \to \psi) \equiv \varphi \wedge \psi$ and Łukasiewicz logic resulted by the extensions of IMTL logic and BL after the axioms (DIV) and (INV) were included, respectively. The following table (4.1) is a summary of the axiom acronyms introduced so far:

Table 4.1: acronyms and axioms

Acronym	*Axiom*
PL	$(\varphi \to \psi) \vee (\psi \to \varphi)$
INV	$\neg\neg\varphi \to \varphi$
WNM	$((\varphi \& \psi) \to 0) \vee ((\varphi \wedge \psi) \to (\varphi \& \psi))$
DIV	$\varphi \& (\varphi \to \psi) \equiv \varphi \wedge \psi$

4.2 ML-algebra

As stated before, the algebraic counterpart of ML logic is the class of residuated lattices. Some authors prefer to use the bounded integral commutative lattice terminology (e.g. [GMO04]), but once both structures are essentially the same [GCDK07], we consider the following definition (based on [GMO04] and [Höh96]) for a ML-algebra:

Definition 4.1 $\mathcal{A} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *is a ML-algebra if:*

1. $\langle L, \wedge, \vee\rangle$ *is a bounded lattice with order* $\leq_{\mathcal{L}}$*, top element* 1 *and bottom* 0*;*
2. $\langle L, *, 1\rangle$ *is a commutative semigroup with unit element 1;*
3. $*$ *and* $\Rightarrow$ *form an adjoint pair (i.e.* $y \leq_{\mathcal{L}} x \Rightarrow z$ *iff* $x * y \leq_{\mathcal{L}} z$ *, for all* $x, y, z \in L$*).*

Thus, ML-algebras are residuated lattices.

The unary operation $\neg$, defined by $\neg x =_{def} x \to 0$, is called **negation**. And, we can also define the operation: $x \oplus y =_{def} \neg(\neg x * \neg y)$, called **disjunction**.

Many classes of ML-algebras can be obtained by including some properties which, in the end, means that refinements of ML-algebras, suitable for fuzzy logics, are defined by the addition of specific axioms to the Monoidal Logic. In this way, we have the following classes of residuated lattices, resulted from the following definitions:

Definition 4.2 *A ML-algebra $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is*

(INV) dualizing iff $\neg\neg x = x$, for all $x \in L$;

(G) idempotent iff $x * x = x$, for all $x \in L$;

(PRL) prelinear iff $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$, for all $x, y \in L$;

(DIV) divisible iff $x \wedge y = x * (x \Rightarrow y)$, for all $x, y \in L$;

(S) weakly contracting iff $x \wedge \neg x = 0$, for all $x \in L$;

($\prod$) weakly cancellative iff $\neg\neg x \leq (x \Rightarrow (x * y)) \Rightarrow y$, for all $x, y \in L$.

These classes can receive special names, such as the Dualizing ML-algebra, which is called AMALL-algebra, for short (standing for the affine multiplicative additive fragment of Linear logic). Other classes are definitely known in the literature, for instance, the second one, which is also called Prelinear ML-algebra, is the MTL-algebra (introduced in [EG01]), where $*$ is the left-continuous t-norm and $\Rightarrow_*$ is the *residumm* of $*$, forming the adjoint pair $(*, \Rightarrow_*)$.

Table 4.2: Classes of residuated lattices and their respective names

Name	*Class of Residuated Lattices*	*Property included*
AMALL-algebra	Dualizing ML-algebra	ML + (INV)
MTL-algebra	Prelinear ML-algebra	ML + (PRL)
IMTL-algebra	Dualizing MTL-algebra	MTL + (INV)
SMTL-algebra	Weakly contracting MTL-algebra	MTL + (S)
BL-algebra	Divisible MTL-algebra	MTL + (DIV)
Ł-algebra	Dualizing BL-algebra	BL + (INV)
G-algebra	Idempotent BL-algebra	BL + (G)
$\prod$-algebra	Weakly contracting and weakly cancellative BL-algebra	BL + (S) + ($\prod$)

Different classes of ML-algebras (some of which are very known in the literature) can be obtained whenever we add one of the above properties. Observe table (4.2) above.

The most famous members of these classes are: MTL-algebra, IMTL-algebra, SMTL-algebra, each one having the left-continuous t-norm $*$ and its *residuum* $\Rightarrow_*$, and BL-algebra, Ł-algebra, G-algebra, $\prod$-algebra, whose t-norms are continuous with the correspondent *residua*. More specifically, the continuous t-norm $*$ of Ł-algebra, G-algebra and $\prod$-algebra are the Łukasiewicz, Gödel and Product t-norms, respectively.

Note that we write $\vDash_L A$ iff A is valid in all L-algebras.

The understanding of this chapter (and undoubtedly the others) is essential to the following one, where we are going to present the interval constructor on ML-algebras. Actually, the majority of definitions and propositions given throughout the previous chapters is going to be used (directly or indirectly) during next chapter.

Chapter 5

The Interval Constructor on ML

A more complete diagram of the relationships between fuzzy logics was given in [GMO04], shown on figure 5.1 below:

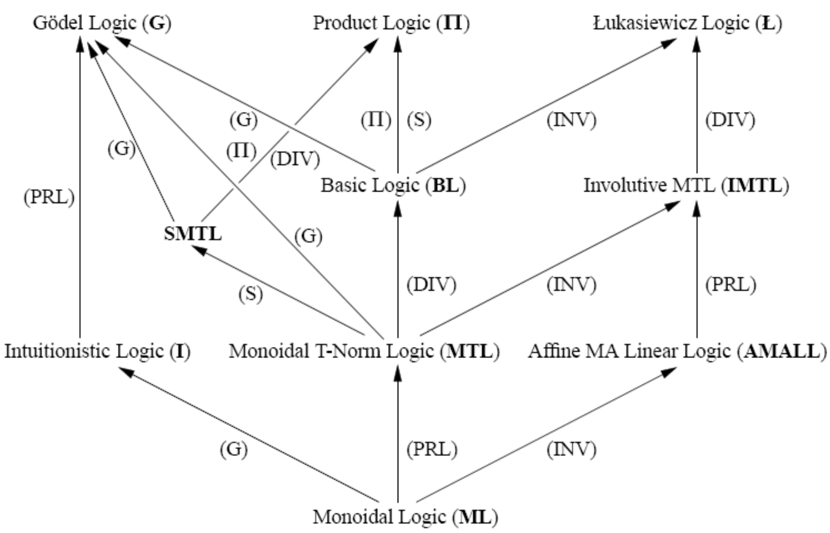

Figure 5.1: Relationships between fuzzy logics.

We can observe other logics were included, as well as other axioms. For example, Gödel Logic (G), obtained after the inclusion of axiom $(G) : \varphi \to (\varphi \& \varphi)$ on BL, and SMTL logic, an extension of MTL after the inclusion of axiom $(S) : \varphi \wedge \neg\varphi \to 0$.

Based on this diagram and the one given on chapter 4 (figure 4.1), we will draw new diagrams to show the relationships between some subclasses of ML-algebras and the interval constructor $\mathbb{I}$, in order to show the subclasses which are preserved by the interval constructor.

5.1 The Interval Constructor on ML-algebras

On the previous chapter, we saw that a ML-algebra $(L, \wedge, \vee, *, \Rightarrow, 0, 1)$ is a residuated lattice such that $(*, \Rightarrow)$ form an adjoint pair. Next, we will prove that the class of ML-algebra is preserved by the interval constructor. Formally:

Proposition 5.1 *Let* $\mathcal{A} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be a ML-algebra. Then* $\mathbb{I}[\mathcal{A}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also a ML-algebra.*

Proof: By proposition 3.10, $\mathbb{I}[\mathcal{L}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ is a residuated lattice. Thus, by definition of ML-Algebra it only remains to prove that $\mathbb{I}[*]$ and $\mathbb{I}[\Rightarrow]$ form an adjoint pair, i.e. that $Y \leq_{\mathbb{I}} X\ \mathbb{I}[\Rightarrow]\ Z$ iff $X\ \mathbb{I}[*]\ Y \leq_{\mathbb{I}} Z$, for all $X, Y, Z \in \mathbb{I}[L]$. So, $X\ \mathbb{I}[*]\ Y \leq_{\mathbb{I}} Z$ iff $Y \in \{U \in \mathbb{I}[L] : X\ \mathbb{I}[*]\ U \leq_{\mathbb{I}} Z\}$ iff $Y \leq_{\mathbb{I}} sup\{U \in \mathbb{I}[L] : X\ \mathbb{I}[*]\ U \leq_{\mathbb{I}} Z\}$ iff $Y \leq_{\mathbb{I}} X\ \mathbb{I}[\Rightarrow]\ Z$. ■

The interval constructor $\mathbb{I}$, on some subclasses of ML-algebras, can also be done by the addition of some properties to certain classes of residuated lattices. In fact, the different classes of residuated lattices, seen on the chapter before, are ML-algebras (or other subclasses of them) which satisfy a certain property (i.e., involution, prelinearity, divisibility, etc.). So, for instance, if a ML-algebra satisfies the dualizing property (INV), we say it is an AMALL-algebra. And the correspondent algebra with the interval constructor is obtained as follows:

Proposition 5.2 *Let* $\mathcal{AM} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be an AMALL-algebra. Then,* $\mathbb{I}[\mathcal{AM}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also an AMALL-algebra.*

Proof: By proposition 5.1, $\mathbb{I}[\mathcal{AM}]$ is already a ML-algebra. Then, by the definition of AMALL-algebra, it only remains to prove that: $(X\ \mathbb{I}[\Rightarrow]\ [0,0])\ \mathbb{I}[\Rightarrow]\ [0,0] = X$, for all $X \in \mathbb{I}[\mathcal{L}]$.

By the definition of $\mathbb{I}[\Rightarrow]$:

$$\begin{aligned}
(X\ \mathbb{I}[\Rightarrow]\ [0,0])\ \mathbb{I}[\Rightarrow]\ [0,0] &= ([\overline{x} \Rightarrow 0, \underline{x} \Rightarrow 0])\ \mathbb{I}[\Rightarrow]\ [0,0] \\
&= [(\underline{x} \Rightarrow 0) \Rightarrow 0, (\overline{x} \Rightarrow 0) \Rightarrow 0] \\
&= [\underline{x}, \overline{x}] \\
&= X. \blacksquare
\end{aligned}$$

We will start drawing a figure to notice the relationships between the classes of ML-algebras according to the propositions presented. The first figure is shown below:

$$\begin{tikzcd}
 & \underset{\text{AMALL- algebra}}{\mathbb{I}[\mathcal{AM}]} \\
\underset{}{\mathbb{I}[\mathcal{A}]}\ \text{ML-algebra} \arrow[ur, "\text{Proposition 5.2}"] &
\end{tikzcd}$$

Figure 5.2: $\mathbb{I}[\mathcal{AM}]$; a dualizing $\mathbb{I}[\mathcal{A}]$.

Hence, the dualizing condition holds, which means $\mathbb{I}[\mathcal{AM}]$ is an AMALL-algebra (observe figure 5.2 above). Besides, by propositions 5.2 and 3.10, and observing table 4.2, we may conclude the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \arrow[r] \arrow[d] & \mathcal{AM} \arrow[d] \\
\underset{\text{ML-algebra}}{\mathbb{I}[\mathcal{A}]} \arrow[r] & \underset{\text{AMALL- algebra}}{\mathbb{I}[\mathcal{AM}]}
\end{tikzcd}$$

Figure 5.3: $\mathbb{I}[\mathcal{AM}]$; a dualizing $\mathbb{I}[\mathcal{A}]$.

Using a similar idea, if an AMALL-algebra satisfies the following property (PRL): $(x \Rightarrow y) \vee (y \Rightarrow x)$, we say it is an IMTL-algebra. Then:

Proposition 5.3 *Let* $\mathcal{IM} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be an IMTL-algebra. Then,* $\mathbb{I}[\mathcal{IM}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also an IMTL-algebra.*

Proof: By proposition 5.2, $\mathbb{I}[\mathcal{IM}]$ is already an AMALL-algebra. So, it only remains to prove that $X\ \mathbb{I}[\Rightarrow]\ Y \sqcup_{\mathbb{I}} Y\ \mathbb{I}[\Rightarrow]\ X = [1,1]$.

$$\begin{aligned}
X\ \mathbb{I}[\Rightarrow]\ Y \sqcup_{\mathbb{I}} Y\ \mathbb{I}[\Rightarrow]\ X &= [\overline{x} \Rightarrow \underline{y}, \underline{x} \Rightarrow \overline{y}] \sqcup_{\mathbb{I}} [\overline{y} \Rightarrow \underline{x}, \underline{y} \Rightarrow \overline{x}] \\
&= [(\overline{x} \Rightarrow \underline{y}) \vee (\overline{y} \Rightarrow \underline{x}), (\underline{x} \Rightarrow \overline{y}) \vee (\underline{y} \Rightarrow \overline{x})] \\
&= [((\overline{x} \Rightarrow \underline{y}) \vee (\overline{y} \Rightarrow \underline{x})) \wedge ((\underline{x} \Rightarrow \overline{y}) \vee (\underline{y} \Rightarrow \overline{x})), \\
&\quad ((\overline{x} \Rightarrow \underline{y}) \vee (\overline{y} \Rightarrow \underline{x})) \vee ((\underline{x} \Rightarrow \overline{y}) \vee (\underline{y} \Rightarrow \overline{x}))] \\
&= [((\overline{x} \Rightarrow \underline{y}) \wedge (\underline{x} \Rightarrow \overline{y})) \vee ((\overline{y} \Rightarrow \underline{x}) \wedge (\underline{x} \Rightarrow \overline{y})) \vee \\
&\quad ((\overline{x} \Rightarrow \underline{y}) \wedge (\underline{y} \Rightarrow \overline{x})) \vee ((\overline{y} \Rightarrow \underline{x}) \wedge (\underline{y} \Rightarrow \overline{x})), \\
&\quad ((\overline{x} \Rightarrow \underline{y}) \vee (\underline{y} \Rightarrow \overline{x})) \vee ((\underline{x} \Rightarrow \overline{y}) \vee (\overline{y} \Rightarrow \underline{x}))] \\
&= [((\overline{x} \Rightarrow \underline{y}) \wedge (\underline{x} \Rightarrow \overline{y})) \vee 1 \vee 1 \vee ((\overline{y} \Rightarrow \underline{x}) \wedge (\underline{y} \Rightarrow \overline{x})), \\
&\quad 1 \wedge 1] \\
&= [1,1]. \blacksquare
\end{aligned}$$

We can continue drawing figure 5.2 by adding proposition 5.3. In this way we have:

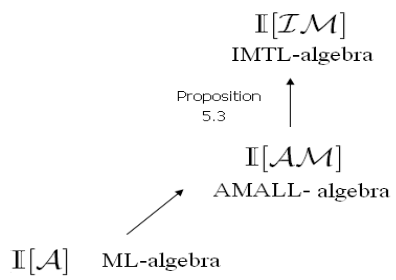

Figure 5.4: $\mathbb{I}[\mathcal{IM}]$; a prelinear $\mathbb{I}[\mathcal{AM}]$.

It is clear $\mathbb{I}[\mathcal{IM}]$ is an IMTL-algebra. Besides, by propositions 5.3 and 3.10, and observing table 4.2, we may conclude diagram 5.5 commutes.

If a ML-algebra satisfies the prelinearity (PRL) property, we say it is a MTL-algebra. Then:

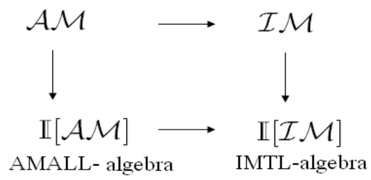

Figure 5.5: $\mathbb{I}[\mathcal{AM}]$; a dualizing $\mathbb{I}[\mathcal{A}]$.

Proposition 5.4 *Let $\mathcal{M} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a MTL-algebra. Then, $\mathbb{I}[\mathcal{M}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ is also a MTL-algebra.*

Proof: By proposition 5.1, $\mathbb{I}[\mathcal{M}]$ is a ML algebra, so according to the definition of MTL-algebra it would only remain to prove that: $X\ \mathbb{I}[\Rightarrow]\ Y \sqcup_{\mathbb{I}} Y\ \mathbb{I}[\Rightarrow]\ X = [1,1]$. But this proof is analogous to the one on the previous proposition (5.3). So, $\mathbb{I}[\mathcal{M}]$ is a MTL-algebra indeed. ■

Including proposition 5.4, we have a new figure:

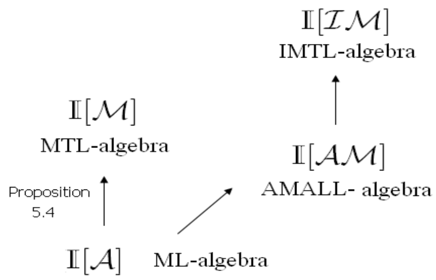

Figure 5.6: $\mathbb{I}[\mathcal{M}]$; a prelinear $\mathbb{I}[\mathcal{A}]$.

Observe that we can also use MTL-algebra $\mathbb{I}[\mathcal{M}]$ to obtain an IMTL-algebra $\mathbb{I}[\mathcal{IM}]$. Analyzing figure 5.1 more carefully, we can see the Involutive MTL is obtained via MTL after the addition of the (INV) axiom. So, including proposition 5.5 given below, we have figure 5.7, showing that MTL-algebra plus property (INV) is an IMTL-algebra.

Proposition 5.5 *Let $\mathcal{IM} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be an IMTL-algebra. Then, $\mathbb{I}[\mathcal{IM}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ is also an IMTL-algebra.*

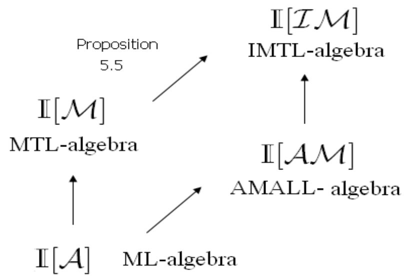

Figure 5.7: $\mathbb{I}[\mathcal{IM}]$; an involutive $\mathbb{I}[\mathcal{M}]$.

Proof: By proposition 5.4, $\mathbb{I}[\mathcal{M}]$ is a MTL algebra, so, according to the definition of IMTL-algebra, it would only remain to prove that: $(X\ \mathbb{I}[\Rightarrow]\ [0,0])\ \mathbb{I}[\Rightarrow]\ [0,0] = X$ holds. But this proof is analogous to the one on proposition 5.2. Therefore, $\mathbb{I}[\mathcal{M}]$ is an IMTL-algebra indeed. ■

Actually, by propositions 5.5 and 3.10, and observing table 4.2, we may conclude the following diagram commutes:

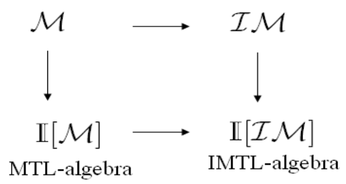

Figure 5.8: $\mathbb{I}[\mathcal{AM}]$; a dualizing $\mathbb{I}[\mathcal{A}]$.

If a MTL-algebra satisfies the weakly contracting property $(S) : X \sqcap_{\mathbb{I}} (X\ \mathbb{I}[\Rightarrow]\ [0,0]) = [0,0]$, we say it is a SMTL-algebra. Then:

Proposition 5.6 *Let* $\mathcal{SM} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be a SMTL-algebra. Then,* $\mathbb{I}[\mathcal{SM}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also a SMTL-algebra.*

Proof: Considering proposition 5.4, we conclude $\mathbb{I}[\mathcal{SM}]$ is a MTL-algebra. And, by the definition of SMTL-algebra, we need to prove that: $X \sqcap_{\mathbb{I}} (X\ \mathbb{I}[\Rightarrow]\ [0,0]) = [0,0]$, for all $X \in \mathbb{I}[\mathcal{L}]$. Hence:

$$
\begin{aligned}
X \sqcap_{\mathbb{I}} (X\ \mathbb{I}[\Rightarrow]\ [0,0]) &= [\underline{x}, \overline{x}]\ \sqcap_{\mathbb{I}} [\overline{x} \Rightarrow 0, \underline{x} \Rightarrow 0] \\
&= [\underline{x} \wedge (\overline{x} \Rightarrow 0), \overline{x} \wedge (\underline{x} \Rightarrow 0)] \\
&= [(\underline{x} \wedge (\overline{x} \Rightarrow 0)) \wedge (\overline{x} \wedge (\underline{x} \Rightarrow 0)), \\
&\quad (\underline{x} \wedge (\overline{x} \Rightarrow 0)) \vee (\overline{x} \wedge (\underline{x} \Rightarrow 0))] \\
&= [(\underline{x} \wedge (\underline{x} \Rightarrow 0)) \wedge (\overline{x} \wedge (\overline{x} \Rightarrow 0)), \\
&\quad (\underline{x} \vee \overline{x}) \wedge ((\overline{x} \Rightarrow 0) \vee \overline{x}) \wedge (\underline{x} \vee (\underline{x} \Rightarrow 0)) \wedge \\
&\quad ((\overline{x} \Rightarrow 0) \vee (\underline{x} \Rightarrow 0))] \\
&= [0 \wedge 0, (\underline{x} \vee \overline{x}) \wedge 0 \wedge 0 \wedge ((\overline{x} \Rightarrow 0) \vee (\underline{x} \Rightarrow 0))] \\
&= [0, 0]. \blacksquare
\end{aligned}
$$

Therefore, $\mathbb{I}[\mathcal{SM}]$ is a SMTL-algebra. And by propositions 5.6 and 3.10, and loooking at table 4.2, we may conclude the following diagram commutes:

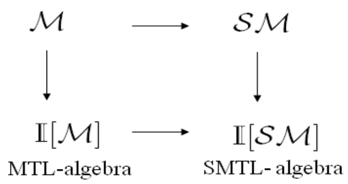

Figure 5.9: $\mathbb{I}[\mathcal{AM}]$; a dualizing $\mathbb{I}[\mathcal{A}]$.

The relationships between ML-algebra ($\mathbb{I}[\mathcal{A}]$), AMALL-algebra ($\mathbb{I}[\mathcal{AM}]$), MTL-algebra ($\mathbb{I}[\mathcal{M}]$), IMTL-algebra ($\mathbb{I}[\mathcal{IM}]$) and SMTL-algebra ($\mathbb{I}[\mathcal{SM}]$) can be seen on figure 5.10.

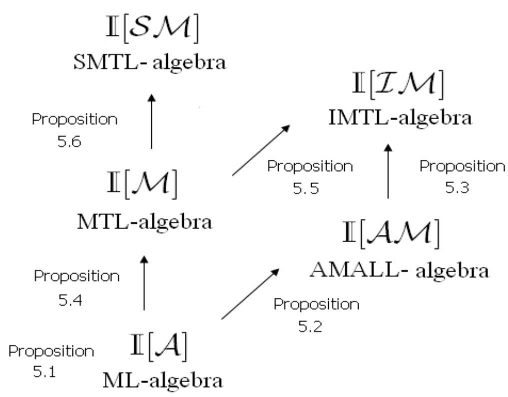

Figure 5.10: Relationships between ML-algebra and its subclasses with the interval constructor $\mathbb{I}$.

Summarizing, it is clear that we obtain:

- $\mathbb{I}[\mathcal{A}]$, from proposition 5.1;
- $\mathbb{I}[\mathcal{AM}]$, from proposition 5.2;
- $\mathbb{I}[\mathcal{IM}]$, from proposition 5.3;
- $\mathbb{I}[\mathcal{M}]$, from proposition 5.4;
- $\mathbb{I}[\mathcal{IM}]$ via $\mathbb{I}[\mathcal{M}]$, from proposition 5.5; and, finally,
- $\mathbb{I}[\mathcal{SM}]$, from proposition 5.6.

We can also construct another diagram considering all the possible ways to achieve classes of ML-algebras by the addition of the dualizing (INV), prelinear (PRL) and weakly contracting (S) properties as showed on figure 5.11.

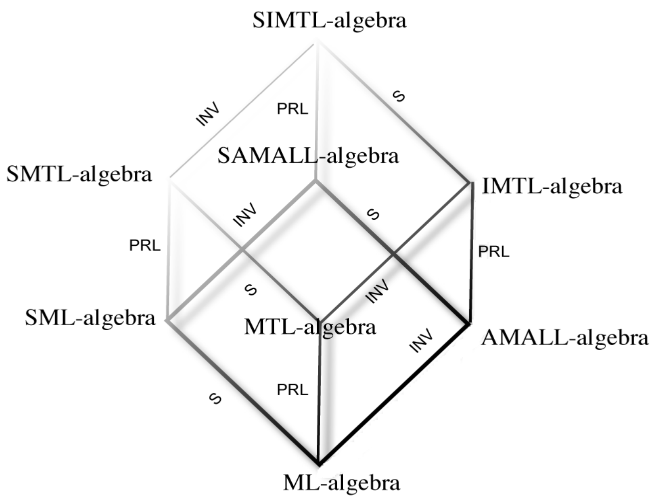

Figure 5.11: Classes of ML-algebras by including properties INV, PRL and S and the interval constructor $\mathbb{I}$.

It is possible to draw some conclusions by observing figure 5.11 more carefully. For instance, it is clear how the following well-known results are achieved: IMTL = AMALL + (PRL) = MTL + (INV) and SMTL = MTL + (S). Besides, possible new ones may be:

- SIMTL = SMTL + (INV) = IMTL + (S),

- SMTL = SML + (PRL),
- SAMALL = AMALL + (S) = SML + (INV) and
- SML = ML + (S).

Clearly, SML-algebra, SAMALL-algebra and SIMTL-algebra would be new classes of ML-algebras after the addition of the properties mentioned previously. Actually, once we have already proved that the interval constructor $\mathbb{I}$ preserves the properties (INV), (PRL) and (S), propositions 5.2, 5.3 and 5.6, then it is easy to see that these classes are closed on the interval constructor.

Chapter 6

Concluding Remarks

This work was a first step to apply the interval constructor $\mathbb{I}$ on classes of ML-algebras. In order to do that, we obtained some interesting results in ML theory itself, for instance, proposition 3.5 and collolary 3.1. Some other related studies, which had been done over the last three years, namely [BS05], [BSCB06], [BCBS07], [CBB01] and [BSCB07] were also relevant to achieve some of our goals.

We had introduced a semantic for fuzzy logic using bounded lattices in [BS05], where we generalized the propositional classic connectives and fuzzy and interval fuzzy connectives using the lattice theory, that is a more general concept. Other results, not entirely connected with this work, but also important, were obtained in [BSCB06] and [BCBS07]. The first one considered a well known generalization of the t-norm notion for arbitrary bounded lattices and introduced two generalizations of the automorphism notion for arbitrary bounded lattices. The latter introduced the t-norm morphism, which is a generalization of the automorphism notion for t-norms on arbitrary bounded lattices. Besides, those generalizations were considered a rich category having t-norms as objects and t-norm morphism as morphism. We proved that putting together that category, a natural transformation introduced there, and the interval constructor on bounded lattice t-norms and t-norm morphisms resulted in an interval category in the sense of [CBB01]. Moreover, in [BSCB07], we proved that the interval category mentioned previously was Cartesian and, whenever its subcategory whose objects were strict t-norms, it was a Cartesian closed category. We showed, as well, that the usual interval construction on lattices was a functor on those categories.

Within this work we used the t-norm generalization mentioned above and extended to other fuzzy connectives: $\mathbb{I}[*], \mathbb{I}[S], \mathbb{I}[\Rightarrow], \mathbb{I}[N]$, the t-norm, t-conorm, implication and negation, respectively. Moreover, we also applied the interval constructor on some classes of bounded lattices, especially the residuated ones, as we mostly dealt with some classes of residuated lattices.

Finally, we used most of those results in order to relate ML-algebras with the interval constructor $\mathbb{I}$, namely, ML-algebra $\mathbb{I}[\mathcal{A}]$, AMALL-algebra $\mathbb{I}[\mathcal{AM}]$, MTL-algebra $\mathbb{I}[\mathcal{M}]$, IMTL-algebra $\mathbb{I}[\mathcal{IM}]$ and SMTL-algebra $\mathbb{I}[\mathcal{SM}]$. In this sense, we proved those algebras are preserved by the interval constructor.

The importance of these results is that each of them determine important classes of fuzzy logics, giving us the possibility to deal with membership degrees represented by intervals in the unit interval $[0, 1]$, which can be used in interval computations allowing an automatic and rigid control of errors (aiming to decrease them), as well as, whenever we deal with many expert's opinions to determine the membership degree without discarding none of their opinions.

For further works we can suggest the addition of other axioms in order to enrich figure 5.1 and obtain other fuzzy logics and consequently other classes of ML-algebras. Another future work would be the inclusion of other logics (even modal logic) within this context. We could also include automorphisms and verify if the classes of ML-algebras are closed on the action of this automorphism (using automorphisms in a similar way as it was done in [BT06]) and relate it to the interval constructor over these classes. Besides , if we pay more attention to the diagram of logics and their relating axioms (figure 5.1), there are lacking arrows (with their respective axioms needed) which would establish other connections between the fuzzy logics. Moreover, it would also be interesting to analyze the behavior of $\mathbb{I}$ on those logics.

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