Some typical classes of t-norms and the 1-Lipschitz condition

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***Abstract*— This paper studies the relation between the satisfaction of the Lipschitz condition by t-norms for constant 1 (1-Lipschitz condition) and some other properties of t-norms. In this sense, we will consider some well know classes of continuous t-norms, such as Archimedean and non Archimedean, and the nilpotent and strict subclasses of Archimedean t-norms. Also will be proved that the unique automorphism which preserves the 1-Lipschitz condition of any t-norm is the identity.**

I. Introduction

There exists uncountable forms of extend the propositional logical connectives to $[0, 1]$ in such a way that on the extremes, i.e. for the boolean values, their behavior be exactly as in the classical connectives. But, the classical connectives satisfies several properties, some of which are desirable that these extensions also satisfies. In [28], Lofti Zadeh used the minimal value to model intersection between fuzzy sets (and therefore also model the conjunction between fuzzy propositions). Lately several other extensions were used, see for example [6], [7], [27]. Nevertheless, in [3], Alsina, Trillas and Valverde used the notion of triangular norms, or simply t-norms, and their dual notion (t-conorm) to model conjunction and disjunction connectives in fuzzy logics, generalizing these several previous fuzzy interpretations. t-norms are binary functions in $[0, 1]$ set which satisfy some natural properties of conjunction such as commutativity and associativity. Triangular norms were originally introduced by Menger in [18] to model the distance in probabilistic metric spaces. But the axiomatic definition of t-norm used today was given by Schweizer and Sklar in [24]. From a t-norm also is possible to obtain canonical fuzzy interpretation for implication and negation connectives. So, each t-norm determine a different set of true formulas (1-tautologies) and false formulas (0-contradictions) and therefore different fuzzy logics [11], [4]. So, t-norms are the basis to study the formal aspects of fuzzy logics, that is the fundamental notion of fuzzy logic in the narrow sense as named in [29].

Automorphisms, in the context of fuzzy set theory, are increasing and bijective functions from $[0, 1]$ into $[0, 1]$. Automorphisms allow to establish when two t-norms are isomorphic [9]; allow to transform t-norms in new t-norms preserving some properties of them and also can be used as generators of t-norms, negations, implications and t-conorms [10], [8].

The Lipschitz condition[1] is stronger than continuity but weaker than derivability and has as main vantage guarantee that iterative processes or equivalently, ordinary differential equations, have a unique solution. This property turn Lipschitz condition reasonable to model dynamic process.

On the other hand, in the convergence of multilayer feedforward neural networks when is used the backpropagation training algorithm, the process to find the weight matrices with statical parameters is dynamic and iterative [26]. So, in a fuzzification of this process, the use of t-norm satisfying the Lipschitz condition would be desired. Clearly, in other way of integration between fuzzy and neural systems, the use of t-norms (and t-conorms) as well as the use iterative and dynamic process will be necessary. Thus, the Lipschitz condition seem be a reasonable requirement for t-norms and their dual t-conorms in fuzzy networks.

In fuzzy logic in the narrow sense were studied several classes of t-norms, i.e. subsets of t-norms which satisfy some conditions or properties. For example, the classes of continu-

[1]Rudolf Lipschitz (1832-1903) published this condition in 1876. Today's formulation of Lipschitz condition appears by first time in p.207 of [23].

ous t-norms, Archimedean t-norms, etc. Still, in spite of the Lipschitz condition be relevant in mathematics, particularly in differential equation theory, for t-norms [2] this conditions has not been very studied. Particularly, the 1-Lipschitz condition for t-norms, i.e. usual Lipschitz condition [23], [12] where the constant is 1, has been considered in some papers such as [24], [20], [25], [2]. In fact [24], [20], [25] proves that 1-Lipschitz t-norms are exactly the t-norms which also are copulas and also that a continuous Archimedean t-norm is 1-Lipschitz if and only has a convex additive generator. In [2] was pointed as open problem the miss of a characterization of the class of t-norms satisfying the k-Lipschitz condition. However, Andrea Mesiarová in a recent paper [19], answered satisfactorily this problem based on a full characterization of their additive generators.

In this paper[2] we will try to determine which t-norms of the class of continuous Archimedean t-norms (and their nilpotent and strict subclasses) satisfies the 1-Lipschitz condition. In this sense, we will prove that the unique nilpotent t-norm satisfying this condition is the Lukasiewicz. Also proved that the product t-norm (the fundamental strict t-norm) satisfy the 1-Lipschitz condition and argument why we belief that it is the unique t-norm in this class satisfying the condition. Thus, since each continuous Archimedean t-norm or is strict or is nilpotent and 1-Lipschitz condition implies in continuity, we can conjecture with strong evidences that the unique two Archimedean t-norms satisfying the 1-Lipschitz condition are the Lukasiewicz and the product. But, we also shows that the family of Dubois-Prade t-norms, a subclass of continuous but not Archimedean class of t-norms, also satisfy this condition. As corollary of our results we will also prove that the unique automorphism which preserve the 1-Lipschitz condition of all t-norms is the identity. However, this not implies that there not exists another automorphism preserving the general Lipschitz condition, but for these automorphisms the constant of the Lipschitz condition of some t-norm would must to change. In fact, it is well known (see for example [13]) that any concave automorphism preserve the Lipschitz condition.

II. T-NORMS

A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a **triangular norm**, t-norm in short, if it satisfy the follow properties:

1) Symmetry: for each $x, y \in [0,1]$, $T(x,y) = T(y,x)$,
2) Associativity: for each $x,y,z \in [0,1]$, $T(x,T(y,z)) = T(T(x,y),z)$,
3) Monotonicity: for each $x_1,y_1,x_2,y_2 \in [0,1]$ if $x_1 \leq x_2$ and $y_1 \leq y_2$ then $T(x_1,y_1) \leq T(x_2,y_2)$ and
4) One identity: for each $x \in [0,1]$, $T(x,1) = x$.

These properties of t-norms are sufficient to guarantee that each t-norm generalize the classical conjunction when the values are the boolean ones, i.e. $T(0,0) = 0$, $T(0,1) = 0$, $T(1,0) = 0$ and $T(1,1) = 1$ for each t-norm.

Some basic t-norms are:

1) Gödel or minimum: $T_G(x,y) = \min\{x,y\}$
2) Lukasiewicz: $T_L(x,y) = \max\{x+y-1,0\}$
3) Product: $T_P(x,y) = xy$
4) Weak:

$$T_{Weak}(x,y) = \begin{cases} \min\{x,y\} & \text{, if } \max\{x,y\} = 1 \\ 0 & \text{, otherwise} \end{cases}$$

5) Hamacher: For each $\gamma \geq 0$ define

$$T_{H,\gamma}(x,y) = \frac{xy}{\gamma + (1-\gamma)(x+y-xy)}$$

6) Dubois and Prade: For each $\alpha \in [0,1]$ define

$$T_\alpha(x,y) = \frac{xy}{\max\{x,y,\alpha\}}$$

Notice that if $\alpha = 0$ then $T_\alpha = T_G$ and if $\alpha = 1$ then $T_\alpha = T_P$.

A. Classes of t-norms

t-norms can be divided in classes. In the follow we will describe the most usual of them and that also are considered in this study.

Notice that in some t-norms there exists elements $x \neq 0$ such that for some other element $y \neq 0$, $T(x,y) = 0$, in these case x (and y also) is called **zero divisor**. For example, $T_{Weak}(0.5, 0.5) = 0$ and therefore 0.5 is a zero divisor, in the hold for T_{Weak} each $x \in (0,1)$ is a zero divisor. So, if a t-norm has not zero divisor and $T(x,y) = 0$ then $x = 0$ or $y = 0$. A t-norm is **continuous** if it is continuous in the usual topology of $[0,1]$ (and $[0,1] \times [0,1]$). A t-norm T is called **Archimedean** if for each $x,y \in (0,1)$ there exists a positive integer n such that $T^n(x) < y$, where

$$T^1(x) = T(x,x) \text{ and } T^{k+1}(x) = T(x, T^k(x)).$$

A continuous t-norm is Archimedean iff for each $x \in (0,1)$, $T(x,x) < x$. A continuous Archimedean t-norm which has at least one zero divisor is called **nilpotent** and is called **strict** otherwise. It implies in that a t-norm T is strict iff for each $x,y,z \in [0,1]$ such that $x < y$ and $0 < z$, $T(x,z) < T(y,z)$ [17], [21]. Clearly, T_P, $T_{H,\gamma}$ and $T_{DP,\alpha}$ are strict whereas T_L, T_G and T_{Weak} are not strict. A t-norm is **idempotent** iff $T(x,x) = x$ for each $x \in [0,1]$, for example T_G.

Thus, we can classified the t-norms as continuous and not-continuous, the continuous t-norms as Archimedean and as not Archimedean, and the continuous Archimedean t-norms as strict or as nilpotent.

A t-norm T satisfy the **1-Lipschitz condition**[2] if for each $x,y,z \in [0,1]$,

$$\mid T(x,z) - T(y,z) \mid \leq \mid x - y \mid . \tag{1}$$

More generical t-norm T satisfy the **k-Lipschitz condition** for a constant $k > 0$ [19] if for each $w,x,y,z \in [0,1]$,

$$\mid T(w,x) - T(y,z) \mid \leq k(\mid w-y \mid + \mid x-z \mid). \tag{2}$$

[2]This work was also submitted to CLEI´2006

Notice that,

1) T satisfy (1) iff T satisfy (2) for $k = 1$,
2) Because one identity property, there not exists t-norms satisfying (2) for $k < 1$.
3) Let $m \geq 1$. If T satisfy (2) for $k = m$ then also satisfy (2) for any $k > m$.

Therefore, 1 is the more strong Lipschitz condition possible for t-norms. However, even is few the knowledge on the class of t-norm satisfying the 1-Lipschitz condition. Still less is know the relationship between the classes previously seen in this subsection with this condition.

III. Automorphisms

Bijective and monotonic functions[3] $\rho : [0,1] \longrightarrow [0,1]$ are called **automorphisms** [17], [21]. Examples of automorphisms are the exponents:

$$e^r(x) = x^r$$

for any $r \in \mathbb{R}^+$. Notice, that $(e^r)^{-1} = e^{\frac{1}{r}}$, and therefore also is an automorphism.

Some well known facts on automorphisms (see for example [13]) are expressed in the following propositions. In particular the first of them state that the definitions of automorphisms given by [17], [21] and [8] are equivalents.

Proposition 3.1: Let $\rho : [0,1] \longrightarrow [0,1]$ be a monotonic function. Then ρ is bijective iff $\rho(1) = 1$, $\rho(0) = 0$ and ρ is continuous and strictly increasing.

Proposition 3.2: Let ρ be an automorphism. If $x, y \in (0,1)$ and $x < y$ then $\rho(x) < \rho(y)$.

Proposition 3.3: Let ρ_1 and ρ_2 automorphism. Then $\rho_1 \circ \rho_2$ also is an automorphism.

Proposition 3.4: Let ρ be an automorphism. Then ρ^{-1} also is an automorphism.

Proposition 3.5: Let ρ_1 and ρ_2 automorphisms. Then $\rho_1 \circ \rho_2$ also is an automorphism and $(\rho_1 \circ \rho_2)^{-1} = \rho_2^{-1} \circ \rho_1^{-1}$.

In fact, $(Aut([0,1], \circ)$ is a group [9].

IV. Automorphism preserving t-norms

Let T be a t-norm and ρ be an automorphism. We said that ρ **preserve** T, if for each $x, y \in [0,1]$, $T(\rho(x), \rho(y)) = \rho(T(x,y))$.

In the following we will provide several results on automorphisms which will be useful in the subsection V-A.

Lemma 4.1: Let ρ be an automorphism. If $\rho(z) < z$ for some $z \in [0,1]$, then there exists $x_0 \in [0,z)$ and $y_0 \in (x_0, z]$ such that $\rho(x_0) = x_0$, for each $a \in (x_0, y_0)$, $\rho(a) < a$ and for each $b \in (a, y_0]$, $a - \rho(a) < b - \rho(b)$.

Proof: Let $x_0 = \sup\{a \in [0,z) : \rho(a) = a\}$. Since $\rho(0) = 0$ and by the completeness of real line, x_0 is well defined. Clearly, by bijectivity of ρ, $\rho(x_0) = x_0$ and by continuity of ρ, $\rho(a) < a$ for each $a \in (x_0, z]$.

Suppose that there not exists such y_0. Then for each $y \in (x_0, z]$ there exists $a \in (x_0, y)$ and $b \in (a, y]$ such that $a - \rho(a) \geq b - \rho(b)$. Since, we can consider y as neighbor as of x_0 when desired, then would must to exists and $\epsilon > 0$ such that for each $a \in (x_0, x_0 + \epsilon)$, $a - \rho(a) \geq x_0 + \epsilon - \rho(x_0 + \epsilon)$. But it is not compatible with the continuity of ρ. ∎

Lemma 4.2: Let ρ be an automorphism. If $\rho(z) < z$ for some $z \in [0,1]$, then there exists $x_1 \in (z, 1]$ and $y_1 \in [z, 1)$ such that $\rho(x_1) = x_1$, for each $a \in (y_1, x_1)$, $\rho(a) < a$ and for each $b \in (a, y_0]$, $b - \rho(b) < a - \rho(a)$.

Proof: Analogous to prove 4.1. ∎

Lemma 4.3: Let ρ be an automorphism. If $z < \rho(z)$ for some $z \in [0,1]$, then there exists $x_0 \in [0,z)$ and $y_0 \in (x_0, z]$ such that $\rho(x_0) = x_0$, for each $a \in (x_0, y_0)$, $a < \rho(a)$ and for each $b \in (a, y_0]$, $\rho(a) - a < \rho(b) - b$.

Proof: Analogous to prove 4.1. ∎

Lemma 4.4: Let ρ be an automorphism. If $z < \rho(z)$ for some $z \in [0,1]$, then there exists $x_1 \in (z, 1]$ and $y_1 \in [z, 1)$ such that $\rho(x_1) = x_1$, for each $a \in (y_1, x_1)$, $a < \rho(a)$ and for each $b \in (a, y_0]$, $\rho(b) - b < \rho(a) - a$.

Proof: Analogous to prove 4.1. ∎

Proposition 4.1: Let ρ be an automorphism and $z \in [0,1]$ such that $\rho(z) < z$. If x_0 and y_0 are as in the lemma 4.1, then for each $a, b \in (x_0, y_0)$ such that $a \neq b$, $\mid \rho(a) - \rho(b) \mid < \mid a - b \mid$.

Proof: Suppose that $\mid \rho(a) - \rho(b) \mid \geq \mid a - b \mid$. If $a < b$ then by monotonicity of ρ, $\rho(a) < \rho(b)$. So, $\mid \rho(a) - \rho(b) \mid = \rho(b) - \rho(a)$ and $\mid a - b \mid = b - a$. Therefore, $b - a \leq \rho(b) - \rho(a)$. Thus, $b - \rho(b) \leq a - \rho(a)$ which is a contradiction with lemma 4.1. ∎

Proposition 4.2: Let ρ be an automorphism and $z \in [0,1]$ such that $\rho(z) < z$. If x_1 and y_1 are as in the lemma 4.2, then for each $a, b \in (y_1, x_1)$ such that $a \neq b$, $\mid \rho(a) - \rho(b) \mid > \mid a - b \mid$.

Proof: Analogously of proposition 4.1. ∎

Proposition 4.3: Let ρ be an automorphism and $z \in [0,1]$ such that $z < \rho(z)$. If x_0 and y_0 are as in the lemma 4.3, then for each $a, b \in (x_0, y_0)$ such that $a \neq b$, $\mid \rho(a) - \rho(b) \mid > \mid a - b \mid$.

Proof: Analogous to proposition 4.1. ∎

Proposition 4.4: Let ρ be an automorphism and $z \in [0,1]$ such that $z < \rho(z)$. If x_1 and y_1 are as in the lemma 4.4, then for each $a, b \in (y_1, x_1)$ such that $a \neq b$, $\mid \rho(a) - \rho(b) \mid < \mid a - b \mid$.

Proof: Analogous to proposition 4.1. ∎

Let $\rho : [0,1] \longrightarrow [0,1]$ be an automorphism and $T : [0,1] \times [0,1] \longrightarrow [0,1]$ be a t-norm. Define $T^\rho : [0,1] \times [0,1]$ by

$$T^\rho(x,y) = \rho^{-1}(T(\rho(x), \rho(y)))$$

T^ρ is a t-norm [17], [21].

[3] In this paper "monotonic functions" means "increasing function".

For example, because $(e^r)^{-1}(x) = e^{\frac{1}{r}}(x) = \sqrt[r]{x}$ for each $r > 0$ and $x \in [0,1]$,

$$\begin{aligned} T_L^{e^3}(x,y) &= (e^3)^{-1}(T_L(e^3(x), e^3(y))) \\ &= \sqrt[3]{T_L(x^3,y^3)} \\ &= \sqrt[3]{\max\{x^3+y^3-1, 0\}} \\ &= \max\{\sqrt[3]{x^3+y^3-1}, 0\} \end{aligned}$$

We said that a class $\mathcal{T}$ of t-norms is **closed under** a class of automorphisms $\mathcal{A}$ if for each $T \in \mathcal{T}$ and each $\rho \in \mathcal{A}$, $T^\rho \in \mathcal{T}$. The more usual classes of t-norms (continuous, Archimedean, whit zero divisors, nilpotent, etc.) are closed under the class of all automorphisms. However it is not holds for the case of t-norms satisfying the 1-Lipschitz condition.

V. Relating classes of t-norms with 1-Lipschitz condition

In this section we will study in which circumstances t-norms of some of classes considered in the subsection II-A satisfies the 1-Lipschitz condition.

Since, as is well known the 1-Lipschitz condition implies in continuity, a necessary condition for a t-norm satisfy the Lipschitz condition is to be continuous.

A. Class of nilpotent t-norms

The Lukasiewicz t-norm is not only a prototypical example of a nilpotent t-norm, but it is well know that each nilpotent t-norms can be reduced via automorphism to it t-norm [16].

Proposition 5.1: A t-norm T is nilpotent iff there exists an automorphism ρ such that $T = T_L^\rho$.

Proof: See [22]. ■

Proposition 5.2: T_L satisfy the 1-Lipschitz condition.

Proof: Let $x, y, z \in [0,1]$.

Case $x+z > 1$ and $y+z > 1$ then

$$\begin{aligned} \mid T_L(x,z) - T_L(y,z) \mid &= \mid (x+z-1) - (y+z-1) \mid \\ &= \mid x - y \mid . \end{aligned}$$

Case $x+z > 1$ and $y+z \leq 1$ then $y < x$ and

$$\begin{aligned} \mid T_L(x,z) - T_L(y,z) \mid &= \mid (x+z-1) - 0 \mid \\ &= x+z-1 \\ &= x+z-1+y-y \\ &= x-y+(y+z-1) \\ &\leq x-y \\ &= \mid x-y \mid . \end{aligned}$$

Case $x+z \leq 1$ and $y+z > 1$ is analogous to previous one.

Case $x+z \leq 1$ and $y+z \leq 1$ then

$$\begin{aligned} \mid T_L(x,z) - T_L(y,z) \mid &= \mid 0 - 0 \mid \\ &= 0 \leq \mid x-y \mid . \end{aligned}$$ ■

Lemma 5.1: Let ρ be an automorphism. T_L^ρ satisfy the 1-Lipschitz condition iff ρ is the identity.

Proof: ($\Rightarrow$) Suppose that ρ is not the identity. Then there exists $z' \in (0,1)$ such that $\rho(z') < z'$ or $z' < \rho(z')$.

If $\rho(z') < z'$ then by lemma 4.1 and 4.2 there exists x_0, y_0, y_1 and x_1 satisfying the conditions of these lemmas. Without loss of generality, we can choice y_0 and y_1 such that $y_0 - x_0 = x_1 - y_1$.

Trivially, if $\rho(z') < z'$ then $z' < \rho^{-1}(z')$ and therefore, by lemmas 4.3 and 4.4, there exists x'_0, y'_0, y'_1 and x'_1 satisfying the conditions of these lemmas. More over, $x_0 = x'_0$, $x_1 = x'_1$, $y_0 = y'_0$, and $y_1 = y'_1$.

Thus, by proposition 4.3, for each $a, b \in (x_0, y_0)$ such that $a \neq b$,

$$\mid \rho^{-1}(a) - \rho^{-1}(b) \mid > \mid a - b \mid . \tag{3}$$

Let $C = y_1 - x_0$, $z = \rho^{-1}(1-C)$, $x = \rho^{-1}(a+C)$ and $y = \rho^{-1}(b+C)$. Thus, $\mid x-y \mid = \mid \rho^{-1}(a+C) - \rho^{-1}(b+C) \mid$.

On the other hand, because $x_1 - y_1 = y_0 - x_0$, $y_1 < a + C < x_1$ and $y_1 < b + C < x_1$. Thus, by proposition 4.4, $\mid \rho^{-1}(a+C) - \rho^{-1}(b+C) \mid < \mid (a+C) - (b+C) \mid = \mid a - b \mid$. Therefore,

$$\mid x - y \mid < \mid a - b \mid . \tag{4}$$

On the other hand,

$$\begin{aligned} a &= a + C - C \\ &= \rho(\rho^{-1}(a+C)) - C \\ &= \rho(x) - C \\ &= \rho(x) + 1 - C - 1 \\ &= \rho(x) + \rho(\rho^{-1}(1-C)) - 1 \\ &= \rho(x) + \rho(z) - 1. \end{aligned} \tag{5}$$

Analogously,

$$b = \rho(y) + \rho(z) - 1. \tag{6}$$

Nevertheless, by hypothesis, T_L^ρ satisfy the 1-Lipschitz condition and therefore $\mid T_L^\rho(x,z) - T_L^\rho(y,z) \mid \leq \mid x - y \mid$. So, by (4),

$$\mid T_L^\rho(x,z) - T_L^\rho(y,z) \mid < \mid a - b \mid . \tag{7}$$

But,

$$\begin{aligned} &\mid T_L^\rho(x,z) - T_L^\rho(y,z) \mid &= \\ &\mid \rho^{-1}(T_L(\rho(x),\rho(z))) - \rho^{-1}(T_L(\rho(y),\rho(z))) \mid &= \\ &\mid \rho^{-1}(\max\{\rho(x)+\rho(z)-1, 0\}) - \rho^{-1}(\max\{\rho(y)+\rho(z)-1\}) \mid &= \\ &\quad \text{(by (5) and (6))} & \\ &\mid \rho^{-1}(\max\{a,0\}) - \rho^{-1}(\max\{b,0\}) \mid &= \\ &\mid \rho^{-1}(a) - \rho^{-1}(b) \mid &> \\ &\mid a - b \mid \text{ (by (3))}, & \end{aligned}$$

which is a contradiction with (7).

The cases when $a, b \in (y_1, x_1)$ (with $\rho(z') < z'$) and when $z' < \rho(z')$ are analogous. So ρ must be the identity.

($\Leftarrow$) Straightforward of proposition 5.2. ■

Theorem 5.1: The unique nilpotent t-norm which satisfy the 1-Lipschitz condition is the Lukasiewicz.

Proof: Straightforward of propositions 5.1 and 5.2 and lemma 5.1. ■

Corollary 5.1: Let $\mathcal{L}$ be the class of t-norms satisfying the 1-Lipschitz condition. The unique automorphism which is closed under the class $\mathcal{L}$ is the identity.

Proof: Straightforward of lemma 5.1. ∎

But it not implies that some large proper subclasses of $\mathcal{L}$ will be closed for a non-trivial class of automorphisms.

B. The class of strict t-norms

Analogously to Lukasiewicz t-norm for nilpotent class of t-norms, the product t-norm is a basis of the class of strict t-norms, in the sense that each strict t-norm can be reduced via automorphism to it t-norm [8], [16].

Proposition 5.3: A t-norm T is strict iff there exists an automorphism ρ such that $T = T_P^\rho$.

Proof: See [1] or [25]. ∎

Proposition 5.4: T_P satisfy the 1-Lipschitz condition.

Proof: Let $x, y, z \in [0, 1]$. Then, trivially,

$$\begin{aligned} \mid T_P(x,z) - T_P(y,z) \mid &= \mid xz - yz \mid \\ &= \mid z(x-y) \mid \\ &= z \mid x - y \mid \\ &\leq \mid x - y \mid . \end{aligned}$$ ∎

In the follows we will prove that a sufficient condition for an automorphism preserve the 1-Lipschitz condition of a strict t-norm T is that it preserve the t-norm.

Proposition 5.5: Let T be an strict t-norm which satisfy the 1-Lipschitz condition and ρ an automorphism which preserve T. Then the strict t-norm T^ρ also satisfy the 1-Lipschitz condition.

Proof: Trivially,

$$\begin{aligned} &\mid T^\rho(x,z) - T^\rho(y,z) \mid &&= \\ &\mid \rho^{-1}(T(\rho(x),\rho(z))) - \rho^{-1}(T(\rho(y),\rho(z))) \mid &&= \\ &\mid \rho^{-1}(\rho(T(x,z)) - \rho^{-1}(\rho(T(y,z))) \mid &&= \\ &\mid T(x,z) - T(y,z) \mid &&\leq \mid x - y \mid . \end{aligned}$$ ∎

Notice that if ρ preserve T trivially $T^\rho = T$. So the proposition 5.5 is trivial, but was proved it to show that these property seem to be essential to prove that T^ρ also satisfy the 1-Lipschitz condition. In other word, the prove of this proposition evidence that the constraint on the automorphism not only is sufficient (as explicitly is put but also is necessary to the automorphism preserve the 1-Lipschitz condition of strict t-norms. Since, by proposition 5.3, for each strict t-norm T there exists an automorphism ρ such that $T = T_P^\rho$, seem that for T satisfy the 1-Lipschitz condition, ρ must preserve product and therefore T would be equal to T_P. So, we conjecture that the unique strict t-norm which satisfy the 1-Lipschitz condition is the product. Therefore, if this conjecture is correct as we belief, the unique archimedean t-norms satisfying the Lipschitz condition are T_L and T_P.

C. Continuous non-Archimedean t-norms

Nevertheless, as will prove, there exists a big subclass of continuous non-Archimedean t-norms satisfying the 1-Lipschitz condition. These class is the Dubois-Prade t-norms (without consider the $\alpha = 1$ i.e. the T_P).

Proposition 5.6: Let $\alpha \in [0, 1)$. Then T_α is a continuous non- Archimedean t-norm.

Proof: Considering that each Dubois-Prade t-norm is an ordinal sum with only one summand namely T_P [14], page 411, and T_P is continuous, then these ordinal sum also is continuous [15], page 422. Trivially, if $\alpha \in [0, 1)$ then $T_\alpha(\alpha, \alpha) = \alpha$ and therefore is not Archimedean. ∎

Proposition 5.7: For each $\alpha \in [0, 1]$, T_α satisfy the 1-Lipschitz condition.

Proof: Let $x, y, z \in [0, 1]$. Then,

$\mid T_\alpha(x,z) - T_\alpha(y,z) \mid = \mid \frac{xz}{\max\{x,z,\alpha\}} - \frac{yz}{\max\{y,z,\alpha\}} \mid$. Thus, case

1) $\max\{x, z, \alpha\} = x$ and $\max\{y, z, \alpha\} = y$ then
$\mid T_\alpha(x,z) - T_\alpha(y,z) \mid = \mid \frac{xz}{x} - \frac{yz}{y} \mid = \mid z - z \mid \leq \mid x - y \mid$
2) $\max\{x, z, \alpha\} = x$ and $\max\{y, z, \alpha\} = z$ then $y < z \leq x$. So,
$\mid T_\alpha(x,z) - T_\alpha(y,z) \mid = \mid \frac{xz}{x} - \frac{yz}{z} \mid = \mid z - y \mid \leq \mid x - y \mid$
3) $\max\{x, z, \alpha\} = x$ and $\max\{y, z, \alpha\} = \alpha$ then $y < \alpha \leq x$ and $z \leq \alpha$. So, $\frac{z}{\alpha} \leq 1$ and $\alpha - y \leq x - y$. Therefore,
$$\begin{aligned} \mid T_\alpha(x,z) - T_\alpha(y,z) \mid &= \mid \tfrac{xz}{x} - \tfrac{yz}{\alpha} \mid \\ &= \mid z - \tfrac{yz}{\alpha} \mid \\ &= \mid \tfrac{z\alpha - yz}{\alpha} \mid \\ &= \mid \tfrac{z}{\alpha}(\alpha - y) \mid \\ &\leq \mid x - y \mid \end{aligned}$$
4) $\max\{x, z, \alpha\} = z$ and $\max\{y, z, \alpha\} = y$ then $x \leq z \leq y$. So,
$\mid T_\alpha(x,z) - T_\alpha(y,z) \mid = \mid \frac{xz}{z} - \frac{yz}{y} \mid = \mid x - z \mid \leq \mid x - y \mid$
5) $\max\{x, z, \alpha\} = z$ and $\max\{y, z, \alpha\} = z$ then
$\mid T_\alpha(x,z) - T_\alpha(y,z) \mid = \mid \frac{xz}{z} - \frac{yz}{z} \mid = \mid x - y \mid$
6) $\max\{x, z, \alpha\} = z$ and $\max\{y, z, \alpha\} = \alpha$ then $\alpha \leq z$ and $z \leq \alpha$. So, $\alpha = z$ therefore idem to previous item.
7) $\max\{x, z, \alpha\} = \alpha$ and $\max\{y, z, \alpha\} = y$ then $x \leq \alpha \leq y$ and $\frac{z}{\alpha} \leq 1$. So,
$$\begin{aligned} \mid T_\alpha(x,z) - T_\alpha(y,z) \mid &= \mid \tfrac{xz}{\alpha} - \tfrac{yz}{y} \mid \\ &= \mid \tfrac{xz}{\alpha} - z \mid \\ &= \mid \tfrac{xz - \alpha z}{\alpha} \mid \\ &= \mid \tfrac{z}{\alpha}(x - \alpha) \mid \\ &\leq \mid x - y \mid \end{aligned}$$
8) $\max\{x, z, \alpha\} = \alpha$ and $\max\{y, z, \alpha\} = z$ then $z = \alpha$ which is idem to item 6.
9) $\max\{x, z, \alpha\} = \alpha$ and $\max\{y, z, \alpha\} = \alpha$ then $\frac{x}{\alpha} \leq 1$. So,
$\mid T_\alpha(x,z) - T_\alpha(y,z) \mid = \mid \frac{xz}{\alpha} - \frac{yz}{\alpha} \mid = \mid \frac{z}{\alpha}(x - y) \mid \leq \mid x - y \mid$

So, T_α satisfy the 1-Lipschitz condition. ∎

Therefore, the Dubois-Prade t-norms (without T_P) is a family of continuous non-Archimedean t-norm which satisfy the Lipschitz condition. But, probably, this family is not the unique t-norms in the class of non-Archimedean t-norms which satisfy the 1-Lipschitz condition.

VI. Final remarks

The corollary 5.1 show that the unique automorphism which preserve the 1-Lipschitz condition of any t-norms is the identity. This fact is not very strange when seem, because the unique automorphism which satisfy the 1-Lipschitz condition ($| \rho(x) - \rho(y) | \leq | x - y |$) is the identity (for see its, it is sufficient to consider the cases when $y = 0$ (and x arbitrary) and $x = 1$ (and y arbitrary)). This result, however isolated in the paper, is be interesting, because the automorphisms are closed for the most of usual classes of t-norms. Notice however that, for the more general case there are several automorphisms (for example concave automorphisms) which preserve the Lipschitz condition, still the constant could be changed. In fact, if ρ is a concave automorphism then T_L^ρ not satisfy the 1-Lipschitz condition (as proved in lemma 5.1) but satisfy the k-Lipschitz condition for some $k > 1$.

But the objective of this paper was to relate the 1-Lipschitz condition with usual classes of t-norms. In this sense, the result obtained are only conclusive with the class of nilpotent t-norms (beyond of non-continuous t-norms, which is obvious). The other two classes analyzed (Strict and non-Archimedean) the result are partial. However, for the the case of strict t-norms class were given evidences to belief that there exists an unique t-norm in the class satisfying the 1-Lipschitz condition. For the other class (non-Archimedean) were found an uncountable family of t-norm satisfying this condition. So, up to least of Lukasiewicz and product t-norms, seem a necessary condition for a t-norm satisfy the 1-Lipschitz condition is that it be continuous and non-Archimedean.

Since the fact of we have a characterization of class $\mathcal{C}$ not implies that we have a characterization of each subclass, the fully characterization provided by Mesiarová, not invalid our result. Thus, for example, Mesiarová characterization not answer which is the isotropy group[4] of the class of t-norms satisfying 1-Lipschitz condition. Moreover, such characterization is given in terms of additive generators without explicitly show their relation with some well know classes of t-norms as is the intention of our partial characterization.

Thus, this paper hope has contribute for a better knowledge of the class of t-norms which satisfy the 1-Lipschitz condition.

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