A Generalized Class of T-norms From a Categorical Point of View

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***Abstract*— Triangular norms or t-norms, in short, and automorphisms are very useful to fuzzy logics in the narrow sense. However, these notions are usually limited to the set $[0,1]$.**

In this paper we will consider a generalization of the t-norm notion for arbitrary bounded lattices as a category, where these generalized t-norms are the objects, and a generalization of automorphism notion as the morphism of the category. We will prove that, this category is Cartesian and a subcategory of it is Cartesian closed. We show that the usual interval t-norms can be seen as a covariant functor for that category.

I. Introduction

Triangular norms were introduced by Karl Menger in [26] with the goal of constructing metric spaces using probabilistic distributions (and therefore values in the interval $[0,1]$), instead of using real numbers, to describe the distance between two elements. Besides, the original proposal is not much restrictive covering t-norms as well as t-conorms. However, only with the work of Berthold Schweizer and Abe Sklar in [30] it was defined t-norms in the axiomatic way used today. In [1], Claudi Alsina, Enric Trillas, and Llorenç Valverde, using t-norms, model the conjunction in fuzzy logics, generalizing several previous fuzzy conjunctions, provided, among others, by Lotfi Zadeh in [35], Richard Bellman and Zadeh in [5] and Ronald Yager in [34]. From a t-norm it is also possible to obtain, canonically, the others propositional connectives [7].

Automorphisms act on t-norms generating in most of the cases a new t-norm. When we see t-norms as semi-groups (see for example [23], [24]) the automorphism yields an isomorphism between t-norms.

On the other hand in fuzzy logic in the narrow sense, it has been very common the use of lattice theory to deal with fuzzy logic in a more general framework, see for example the L-fuzzy set theory [16], BL-algebras of Hájek [18], Brouwerian lattices [33], etc.

In [14], [13] it was generalized the t-norm notion bounded to partially ordered sets, which is a more general structure than the bounded lattice. In [28] it was considered an extension of t-norms for bounded lattice which coincides with the one given by [14], [13]. In this paper we will consider this notion of t-norm for arbitrary bounded lattices.

Categories give a strongly formalized language which is appropriated in order to establish abstract properties of mathematical structures. So, seing the t-norm theory as a category we gain in elegancy and in the comprehension of the general property of t-norms.

A first contribution of this paper is to provide a generalization of the notion of automorphism to bounded lattices. Since automorphism presupposes the use of the same lattice, we also generalize this notion to t-norm morphism which considers different lattices for domain and co-domain. Another contribution is to consider the product, function space and interval lattices constructions to construct t-norms and t-norm morphisms. We also analyze some categorical properties of these constructions, in particular we show that this category is Cartesian and its subcategory, which considers only strict t-norms, is Cartesian closed. We also proved that for both categories the usual interval constructor on lattice [17], [32] is a covariant interval functor, and so, despite not proved here, it is an interval category in the sense of [8].

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II. Lattices

Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be an algebraic structure where L is a nonempty set and $\wedge$ and $\vee$ are binary operations. $\mathbf{L}$ is a **lattice**, if for each $x, y, z \in L$

1) $x \wedge y = y \wedge x$ and $x \vee y = y \vee x$
2) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ and $x \vee (y \vee z) = (x \vee y) \vee z$
3) $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$

In a lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$, if there exist two distinct elements, 0 and 1, such that for each $x \in L$, $x \wedge 1 = x$ and $x \vee 0 = x$ then $\langle L, \wedge, \vee, 1, 0 \rangle$ is said a **bounded lattice**.

Example 2.1: Some examples of bounded lattices:

1) $\mathbf{L}_\top = \langle \{1\}, \wedge, \vee, 1, 1 \rangle$, where $1 \wedge 1 = 1 \vee 1 = 1$.
2) $\mathbf{B} = \langle \mathbb{B}, \wedge, \vee, 1, 0 \rangle$, where $\mathbb{B} = \{0, 1\}$, $\wedge$ and $\vee$ are as in the boolean algebra.
3) $\mathbf{I} = \langle [0,1], \wedge, \vee, 1, 0 \rangle$, where $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.
4) $\mathbf{N} = \langle \mathbb{N}^\top, \wedge, \vee, \top, 0 \rangle$, where $\mathbb{N}$ is the set of natural numbers and
   a) $\mathbb{N}^\top = \mathbb{N} \cup \{\top\}$,
   b) $x \wedge \top = \top \wedge x = x$ and if $x, y \in \mathbb{N}$ then $x \wedge y = \min\{x, y\}$,
   c) $x \vee \top = \top \vee x = \top$ and if $x, y \in \mathbb{N}$ then $x \vee y = \max\{x, y\}$.

As it is well known, each lattice establishes a partial order. Let $\mathbf{L} = \langle L, \wedge, \vee \rangle$ be a lattice. Then $\leq_L \subseteq L \times L$ defined by

$$x \leq_L y \Leftrightarrow x \wedge y = x$$

is a partial order where $\wedge$ coincides with the greatest lower bound (infimum) and $\vee$ with the least upper bound (supremum).

Let $\mathbf{L} = \langle L, \wedge_L, \vee_L, 1_L, 0_L \rangle$ and $\mathbf{M} = \langle M, \wedge_M, \vee_M, 1_M, 0_M \rangle$ be bounded lattices. A function $h : L \longrightarrow M$ is a **lattice homomorphism**[1] if

1) $h(0_L) = 0_M$,
2) $h(1_L) = 1_M$,
3) for each $x, y \in L$ then
   a) $h(x \wedge_L y) = h(x) \wedge_M h(y)$,
   b) $h(x \vee_L y) = h(x) \vee_M h(y)$.

Proposition 2.1: Let $\mathbf{L}$ and $\mathbf{M}$ be bounded lattices. $h : L \longrightarrow M$ is a lattice homomorphism iff $h(0_L) = 0_M$, $h(1_L) = 1_M$, and h is monotonic w.r.t. the lattice orders.

Proof: It is a well known fact. ■

Example 2.2: Let $\mathbf{L}$ be a bounded lattice. Then for each $\alpha \in (0,1)$, the function $h_\alpha : [0,1] \longrightarrow L$ defined by

$$h_\alpha(x) = \begin{cases} 0_L & \text{if } x \leq \alpha \\ 1_L & \text{if } x > \alpha \end{cases}$$

is lattice homomorphism from $\mathbf{I}$ into $\mathbf{L}$.

A. Operators on bounded lattices

Let $\mathbf{L}$ and $\mathbf{M}$ be bounded lattices. The **product** of $\mathbf{L}$ and $\mathbf{M}$, is $\mathbf{L} \times \mathbf{M} = \langle L \times M, \wedge, \vee, (1_L, 1_M), (0_L, 0_M) \rangle$, where $(x_1, x_2) \wedge (y_1, y_2) = (x_1 \wedge_L y_1, x_2 \wedge_M y_2)$ and $(x_1, x_2) \vee (y_1, y_2) = (x_1 \vee_L y_1, x_2 \vee_M y_2)$ is also a bounded lattice.

Let $\mathbf{L}$ be a bounded lattice. The **interval** of $\mathbf{L}$, is $\mathbb{I}\mathbf{L} = \langle IL, \wedge, \vee, [1,1], [0,0] \rangle$ where $IL = \{[\underline{x}, \overline{x}] : \underline{x}, \overline{x} \in L$ and $\underline{x} \leq_L \overline{x}\}$, $[\underline{x}, \overline{x}] \wedge [\underline{y}, \overline{y}] = [\underline{x} \wedge_L \underline{y}, \overline{x} \wedge_L \overline{y}]$ and $[\underline{x}, \overline{x}] \vee [\underline{y}, \overline{y}] = [\underline{x} \vee_L \underline{y}, \overline{x} \vee_L \overline{y}]$ is also a bounded lattice.

The associated order for this lattice agrees with the product order. That is,

$$[\underline{x}, \overline{x}] \leq [\underline{y}, \overline{y}] \text{ iff } \underline{x} \leq_L \underline{y} \text{ and } \overline{x} \leq_L \overline{y} \tag{1}$$

This partial order (1) generalizes a partial order used for the first time by Kulisch and Miranker [22] in the interval mathematics context.

Clearly, bounded lattices are closed under product and interval operators.

III. T-NORMS AND AUTOMORPHISMS ON BOUNDED LATTICES

Let $\mathbf{L}$ be a bounded lattice. A binary operation T on L is a **triangular norm** on $\mathbf{L}$, **t-norm** in short, if for each $v, x, y, z \in L$ the following properties are satisfied:

1) *commutativity:* $T(x,y) = T(y,x)$,
2) *associativity:* $T(x, T(y,z)) = T(T(x,y), z)$,
3) *neutral element:* $T(x, 1) = x$ and
4) *monotonicity:* If $y \leq_L z$ then $T(x,y) \leq_L T(x,z)$.

Notice that for the lattice $\mathbf{I}$ in particular, this notion of t-norm coincides with the usual one. The well known Gödel and weak t-norms (also known by minimum and drastic product t-norm [23]) can be generalized for arbitrary bounded lattice in a natural way. In particular, the Gödel t-norm (T_G) coincides with $\wedge$ itself and the weak t-norm is defined by

$$T_W(x,y) = \begin{cases} 0 & \text{if } x \neq 1 \text{ and } y \neq 1 \\ x \wedge y & \text{otherwise} \end{cases}$$

The t-norm on a same lattice can be partially ordered. Let T_1 and T_2 be t-norms on a bounded lattice $\mathbf{L}$. Then T_1 is **weaker** than T_2 or, equivalently, T_2 is **stronger** than T_1, denoted by $T_1 \leq T_2$ if for each $x, y \in L$, $T_1(x,y) \leq_L T_2(x,y)$.

Proposition 3.1: Let T be a t-norm on a bounded lattice $\mathbf{L}$. Then

$$T_W \leq T \leq T_G$$

Proof: Similar to the classical result (see for example remark 1.5.(i) in [23]). ■

Corollary 3.1: Let T be a t-norm on a bounded lattice $\mathbf{L}$. Then $T(x,y) = 1_L$ iff $x = y = 1_L$.

Proof: Straightforward. ■

Let T be a t-norm on a bounded lattice $\mathbf{L}$. An element $x \in L$ is a **zero divisor** of T if $T(x,y) = 0_L$ for some $y \in L - \{0_L\}$. In case $x \neq 0_L$, x is said a **nontrivial zero divisor** of T. An $x \in L$ is a **nilpotent** element of T if $T(x,x) = 0_L$. A t-norm T with at most one nilpotent element $x \in L - \{0_L, 1_L\}$ is said a **nilpotent t-norm**. A classical result is that a t-norm T is nilpotent iff it has at most one nontrivial zero divisor. A t-norm is **strict** if for each $x \in L - \{0_L, 1_L\}$, $T(x,x) <_L x$. A classical result is that a t-norm T is strict iff it is not nilpotent. So, strict t-norms have no nontrivial zero divisor.

A. T-norm morphisms

Let T_1 and T_2 be t-norms on the bounded lattices $\mathbf{L}$ and $\mathbf{M}$, respectively. A lattice homomorphism $\rho : L \to M$ is a **t-norm morphism** from T_1 into T_2 if for each $x, y \in L$

$$\rho(T_1(x,y)) \leq_M T_2(\rho(x), \rho(y)) \tag{2}$$

Straightforward from the fact that ρ is a lattice morphism, ρ is monotonic.

If there exists a t-norm morphism ρ' from T_2 into T_1 such that $\rho' \circ \rho = Id_L$ and $\rho \circ \rho' = Id_M$, then ρ is a **t-norm isomorphism**. Notice that, there is at most only one t-norm isomorphism between two t-norms, but there can exist several t-norm morphisms. Notice also that for t-norm isomorphism, the inequality (2) is an equality, but not necessarily all t-norm morphisms satisfying $\rho(T_1(x,y)) = T_2(\rho(x), \rho(y))$ are t-norm isomorphisms.

When $\mathbf{L}$ and $\mathbf{M}$ are equal, t-norm isomorphisms are called **automorphisms**. In fact, this notion coincides with the usual notion of automorphism when the lattice is $\mathbf{I}$.

[1] In [12] the term "homomorphism" was used for a lattice not necessarily bounded. For homomorphism preserving bottom and top elements, as considered here, Davey and Priestley use the term $\{0,1\}$-homomorphism.

IV. The Category of Bounded Lattice Strict t-norms

Clearly the composition of two t-norm morphisms is also a t-norm morphism. In fact, let **K**, **L** and **M** be bounded lattices, T_1, T_2 and T_3 be t-norms on **K**, **L**, and **M**, respectively, and ρ_1 and ρ_2 be a morphism between T_1 into T_2 and between T_2 into T_3, respectively. So, $T_3(\rho_2 \circ \rho_1(x), \rho_2 \circ \rho_1(y)) = \rho_2(T_2(\rho_1(x), \rho_1(y))) = \rho_2(\rho_1(T_1(x, y)))$.

Since the composition of functions is associative, then the composition of t-norm morphisms is also associative. Notice that for any bounded lattice **L**, the identity $Id_L(x) = x$ is an automorphism such that for each t-norm on **L**, $Id_L(T(x, y)) = T(Id_L(x), Id_L(y))$.

Thus, considering the t-norm morphism notion as a morphism and t-norms as objects, we have a category, denoted by $\mathcal{T}$.

In the following section we will see some properties of this category and of its subcategory $\mathcal{T}_S$ which has strict t-norms as objects and t-norm morphisms as morphisms.

A. Terminal object

Proposition 4.1: Let $T_\top : \{(1, 1)\} \longrightarrow \{1\}$ defined by $T_\top(1, 1) = 1$. Then $T_\top$ is a strict t-norm on the bounded lattice $\mathbf{L}_\top$.

Proof: Straightforward. ■

Proposition 4.2: Let T be a t-norm on a bounded lattice **L**. Then $\rho_\top : L \rightarrow \{1\}$ defined by $\rho_\top(x) = 1$ is the unique t-norm morphism from T into $T_\top$.

Proof: Straightforward. ■

Thus, $T_\top$ is a terminal object of $\mathcal{T}$ and consequently of $\mathcal{T}_S$.

Proposition 4.3: Let T be a t-norm on a bounded lattice **L**. If there exists a morphism ρ from $T_\top$ into T then T is isomorph to $T_\top$.

Proof: Straightforward. ■

This means that, there is a unique morphism output from $T_\top$.

Corollary 4.1: Neither $\mathcal{T}$ nor $\mathcal{T}_S$ has a generator.

Proof: Straightforward definition of generator, see for example [2]. ■

B. Cartesian Product

Proposition 4.4: Let T_1 and T_2 be t-norms on bounded lattices **L** and **M**, respectively. Then $T_1 \times T_2 : (L \times M)^2 \rightarrow L \times M$ defined by

$$T_1 \times T_2((x_1, x_2), (y_1, y_2)) = (T_1(x_1, y_1), T_2(x_2, y_2))$$

is a t-norm on the bounded lattice $\mathbf{L} \times \mathbf{M}$. Moreover, if T_1 and T_2 are strict then $T_1 \times T_2$ also is.

Proof: Straightforward. ■

Proposition 4.5: Let T_1 and T_2 be t-norms on the bounded lattices **L** and **M**, respectively. Then the usual projections $\pi_1 : L \times M \longrightarrow L$ and $\pi_2 : L \times M \longrightarrow M$ defined by

$$\pi_1(x, y) = x \text{ and } \pi_2(x, y) = y$$

are t-norm morphisms from $T_1 \times T_2$ into T_1 and T_2, respectively.

Proof: As it is well known, π_1 and π_2 are lattice morphisms. So, it only remains to prove that it satisfies Equation 2. Let $(x_1, x_2), (y_1, y_2) \in L \times M$. Then

$$\begin{aligned} \pi_1(T_1 \times T_2((x_1, x_2), (y_1, y_2))) &= \pi_1(T_1(x_1, y_1), T_2(x_2, y_2)) \\ &= T_1(x_1, y_1) \\ &= T_1(\pi_1(x_1, x_2), \pi_1(y_1, y_2)) \end{aligned}$$

The π_2 case is analogous. ■

Next we will prove that $\mathcal{T}$ satisfies the universal property of cartesian product.

Theorem 4.1: Let T, T_1 and T_2 be (strict) t-norms on the bounded lattices **K**, **L** and **M**, respectively. If ρ_1 and ρ_2 are t-norm morphisms from T into T_1 and T_2, respectively, then there exists only one t-norm morphism ρ from T into $T_1 \times T_2$ such that the following diagram commutes:

$$\begin{tikzcd} & T \arrow[dl, "\rho_1"'] \arrow[d, dashed, "\rho"] \arrow[dr, "\rho_2"] & \\ T_1 & T_1 \times T_2 \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & T_2 \end{tikzcd}$$

Proof: Let $\rho : K \longrightarrow L \times M$ be the function:

$$\rho(x) = (\rho_1(x), \rho_2(x))$$

First we will prove that, ρ is a t-norm morphism.

$$\begin{aligned} \rho(T(x, y)) &= (\rho_1(T(x, y)), \rho_2(T(x, y))) \\ &\leq_{L \times M} (T_1(\rho_1(x), \rho_1(y)), T_2(\rho_2(x), \rho_2(y))) \\ &= T_1 \times T_2((\rho_1(x), \rho_2(x)), (\rho_1(y), \rho_2(y)) \\ &= T_1 \times T_2(\rho(x), \rho(y)). \end{aligned}$$

Since, for $i = 1$ and $i = 2$, $\pi_i(\rho(x)) = \pi_i(\rho_1(x), \rho_2(x)) = \rho_i(x)$, then the above diagram commutes.

Suppose that $\rho' : K \longrightarrow L \times M$ is a t-norm morphism which commutes the diagram. Then, $\pi_1(\rho'(T(x, y))) = \rho_1(T(x, y))$ and $\pi_2(\rho'(T(x, y))) = \rho_2(T(x, y))$.

So, $\rho'(T(x, y)) = (\rho_1(T(x, y)), \rho_2(T(x, y))) = \rho(T(x, y))$. Therefore, ρ is the unique t-norm morphism commuting the above diagram. ■

Therefore, we can claim that $\mathcal{T}_S$ is a cartesian category.

C. Exponential

In computing, it could be interesting in some situations when a procedure is an argument of other procedures, and in this case, from a theoretical point of view, we need to deal with higher order functions. In category theory, higher order function is dealt with the notion of an exponent object, which suitably represents the set of morphism from an object to another object in the category [2].

Proposition 4.6: Let T_1 and T_2 be strict t-norms on bounded lattices $\mathbf{L}$ and $\mathbf{M}$, respectively and $[\mathbf{T_1} \to \mathbf{T_2}] = \langle [T_1 \to T_2], \bigwedge, \bigvee, \rho^\top, \rho^\perp \rangle$ where

- $[T_1 \to T_2]$ is the set of all t-norm morphisms from T_1 into T_2;
- $\rho_1 \bigwedge \rho_2(x) = \rho_1(x) \wedge_M \rho_2(x)$, $\forall \rho_1, \rho_2 \in [T_1 \to T_2]$ and $x \in L$;
- $\rho_1 \bigvee \rho_2(x) = \rho_1(x) \vee_M \rho_2(x)$, $\forall \rho_1, \rho_2 \in [T_1 \to T_2]$ and $x \in L$;
- $\rho^\top, \rho^\perp : L \longrightarrow M$ are defined by
  - $\rho_\top(x) = 0_M$ if $x = 0_L$ and $\rho_\top(x) = 1_M$ otherwise;
  - $\rho_\perp(x) = 1_M$ if $x = 1_L$ and $\rho_\top(x) = 0_M$ otherwise.

Then, $[\mathbf{T_1} \to \mathbf{T_2}]$ is a bounded lattice.

Proof: Clearly $\bigwedge$ and $\bigvee$ are commutative and associative. Absorbtion laws: $\rho_1 \bigwedge (\rho_1 \bigvee \rho_2)(x) = \rho_1(x) \wedge_M (\rho_1(x) \vee_M \rho_2(x)) = \rho_1(x)$ and $\rho_1 \bigvee (\rho_1 \bigwedge \rho_2)(x) = \rho_1(x) \vee_M (\rho_1(x) \wedge_M \rho_2(x) = \rho_1(x)$.

So, it only remains to prove that $\rho^\top$ and $\rho^\perp$ are well defined and are the smallest and the greatest t-norm morphisms, respectively.

If $\rho^\top(T_1(x,y)) = 0_M$ then $T_1(x,y) = 0_L$. Since T_1 has no nontrivial zero divisors, $x = 0_L$ or $y = 0_L$. If $x = 0_L$, then $T_2(\rho^\top(x), \rho^\top(y)) = T_2(0_M, \rho^\top(y)) = 0_M$. Analogously, If $y = 0_L$, then $T_2(\rho^\top(x), \rho^\top(y)) = T_2(\rho^\top(x), 0_M) = 0_M$. So, $T_2(\rho^\top(x), \rho^\top(y)) = 0_M$. On the other hand, if $\rho^\top(T_1(x,y)) = 1_M$ then $T_1(x,y) \neq 0_L$ and so $x \neq 0_L$ and $y \neq 0_L$. Therefore, $\rho^\top(x) = \rho^\top(y) = 1_M$ and hence $T_2(\rho^\top(x), \rho^\top(y)) = T_2(1_M, 1_M) = 1_M$.

If $\rho^\perp(T_1(x,y)) = 1_M$ then $T_1(x,y) = 1_L$ and therefore $x = y = 1_L$. So, $T_2(\rho^\perp(x), \rho^\perp(y)) = T_2(1_M, 1_M) = 1_M$. If $\rho^\perp(T_1(x,y)) \neq 1_M$ then $T_1(x,y) \neq 1_M$. Thus, either $x \neq 1_M$ or $y \neq 1_M$ and hence either $\rho^\perp(x) \neq 1_M$ or $\rho^\perp(y) \neq 1_M$. Therefore, by Corollary 3.1, $T_2(\rho^\perp(x), \rho^\perp(y)) \neq 1_M$.

So, $\rho^\top$ and $\rho^\perp$ are well defined, i.e. are t-norm morphisms.

Let ρ be another t-norm morphism from T_1 into T_2, then $(\rho \bigwedge \rho^\top)(0_L) = \rho(0_L) \wedge_M \rho^\top(0_L) = 0_M \wedge_M 0_M = 0_M = \rho(0_L)$. If $x \neq 0_L$ then $(\rho \bigwedge \rho^\top)(x) = \rho(x) \wedge_M \rho^\top(x) = \rho(x) \wedge_M 1_M = \rho(x)$.

Let ρ be another t-norm morphism from T_1 into T_2, then $(\rho \bigvee \rho^\perp)(1_L) = \rho(1_L) \bigvee \rho^\perp(1_L) = 1_M \vee_M 1_M = 1_M = \rho(1_M)$. If $x \neq 1_M$ then $(\rho \bigvee \rho^\perp)(x) = \rho(x) \vee_M \rho^\perp(x) = \rho(x) \vee_M 0_M = \rho(x)$. ∎

Notice that, the lattice order of $[\mathbf{T_1} \to \mathbf{T_2}]$ is defined by

$$\rho_1 \leq \rho_2 \text{ iff } \rho_1(x) \leq_M \rho_2(x) \text{ for each } x \in L$$

where $\leq_M$ is the lattice order of $\mathbf{M}$.

Let T_1 and T_2 be t-norms on bounded lattices $\mathbf{L}$ and $\mathbf{M}$, respectively. The **exponent** of T_1 and T_2 is the function $T_2^{T_1} : [T_1 \to T_2]^2 \longrightarrow [T_1 \to T_2]$ defined by

$$T_2^{T_1}(\rho_1, \rho_2)(x) = T_2(\rho_1(x), \rho_2(x)) \tag{3}$$

Proposition 4.7: Let T_1 and T_2 be t-norms on bounded lattices $\mathbf{L}$ and $\mathbf{M}$, respectively. Then $T_2^{T_1}$ is a t-norm on the bounded lattice $[\mathbf{T_1} \to \mathbf{T_2}]$.

Proof: For each $x \in L$,

- Commutativity: $T_2^{T_1}(\rho_1, \rho_2)(x) = T_2(\rho_1(x), \rho_2(x)) = T_2(\rho_2(x), \rho_1(x)) = T_2^{T_1}(\rho_2, \rho_1)(x)$;
- Associativity: $T_2^{T_1}(\rho_1, T_2^{T_1}(\rho_2, \rho_3))(x) = T_2(\rho_1(x), T_2^{T_1}(\rho_2, \rho_3)(x)) = T_2(\rho_1(x), T_2(\rho_2(x), \rho_3(x))) = T_2(T_2(\rho_1(x), \rho_2(x)), \rho_3(x)) = T_2^{T_1}(T_2^{T_1}(\rho_1, \rho_2), \rho_3)(x)$
- Neutral element: $T_2^{T_1}(\rho, \rho^\top)(x) = T_2(\rho(x), \rho^\top(x)) = T_2(0_M, 0_M) = 0_M = \rho(x)$ if $x = 0_M$. If $x \neq 0_M$, then $T_2^{T_1}(\rho, \rho^\top)(x) = T_2(\rho(x), \rho^\top(x)) = T_2(\rho(x), 1_M) = \rho(x)$. So, $T_2^{T_1}(\rho, \rho^\top)(x) = \rho(x)$ for each $x \in L$.
- Monotonicity: If $\rho_1 \leq \rho_2$ then $\rho_1(x) \leq_M \rho_2(x)$ for each $x \in L$. So, by monotonicity of T_2, $T_2(\rho(x), \rho_1(x)) \leq_M T_2(\rho(x), \rho_2(x))$ and therefore, $T_2^{T_1}(\rho, \rho_1)(x) \leq_M T_2^{T_1}(\rho, \rho_2)(x)$. So, $T_2^{T_1}(\rho, \rho_1) \leq T_2^{T_1}(\rho, \rho_2)$. ∎

Proposition 4.8: Let T_1 and T_2 be t-norms on the bounded lattices $\mathbf{L}$ and $\mathbf{M}$, respectively. Then the function $eval : [T_1 \to T_2] \times L \longrightarrow M$ defined by $eval(\rho, x) = \rho(x)$, is a t-norm morphism from $T_2^{T_1} \times T_1$ into T_2.

Proof: First we need to prove that $eval$ is a lattice homomorphism, to do that we will consider the proposition 2.1 and, finally, that it is a t-norm morphism.

- $eval(\rho^\perp, 0_L) = \rho^\perp(0_L) = 0_M$,
- $eval(\rho^\top, 1_L) = \rho^\top(1_L) = 1_M$,
- If $(\rho, x) \leq (\rho', y)$ then $\rho \leq \rho'$ and $x \leq_L y$. So, $\rho(x) \leq_M \rho(y) \leq_M \rho'(y)$ and therefore, $eval(\rho, x) \leq_M eval(\rho', y)$,
- $eval(T_2^{T_1} \times T_1((\rho, x), (\rho', y)) = eval(T_2^{T_1}(\rho, \rho'), T_1(x,y)) = T_2^{T_1}(\rho, \rho')(T_1(x,y)) = T_2(\rho(T_1(x,y)), \rho'(T_1(x,y))) = T_2(T_2(\rho(x), \rho(y)), T_2(\rho'(x), \rho'(y))) = T_2(T_2(\rho(x), \rho'(y)), T_2(\rho'(x), \rho(y))) \leq_M T_2(\rho(x), \rho'(y)) = T_2(eval(\rho, x), eval(\rho', y))$. ∎

Theorem 4.2: Let T, T_1 and T_2 be t-norms on the bounded lattices $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{M}$, respectively. If ρ is a t-norm morphism from $T \times T_1$ into T_2 then there exists a unique t-norm morphism ρ' from T into $T_2^{T_1}$ such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[d, dashed, "\rho'"'] & T \times T_1 \arrow[r, "\rho"] \arrow[d, dashed, "\rho' \times Id_L"'] & T_2 \\
T_2^{T_1} & T_2^{T_1} \times T_1 \arrow[ur, "eval"'] &
\end{tikzcd}$$

Proof: Let $\rho' : K \longrightarrow [L \rightarrow M]$ be the function defined by $\rho'(x) = \rho_x$ where $\rho_x(y) = \rho(x, y)$. First we will prove that ρ' is the unique lattice morphism such that the above diagram commutes for the underlying lattices.

$$\begin{aligned} eval(\rho' \times Id_L)(x, y) &= eval(\rho'(x), Id_L(y)) \\ &= eval(\rho_x, y) \\ &= \rho(x, y) \end{aligned}$$

If $\widetilde{\rho}$ is another lattice morphism commuting the above diagram then, $eval(\widetilde{\rho}(x), y) = \widetilde{\rho}(x)(y) = \rho(x, y) = \rho'(x)(y)$.

So, only remain to prove that ρ' is a t-norm morphism.

$$\begin{aligned} \rho'(T(x, y))(T_1(x_1, y_1)) &= \rho(T(x, y), T_1(x_1, y_1)) \\ &\leq T_2(\rho(x, x_1), \rho(y, y_1)) \\ &= T_2(\rho_x(x_1), \rho_y(y_1)) \\ &= T_2^{T_1}(\rho_x, \rho_y)(x_1, y_1). \end{aligned}$$

■

Therefore, $\mathcal{T}_S$ is a Cartesian closed category, which is an important property to model the typed λ-calculi [27], [11], [20], [25], [2].

D. Intervals

Interval t-norms have been widely studied in the unit lattice (see for example [36], [15], [4]) as well as in certain classes of lattice (see for example [10], [32], [3]). The main motivation to consider interval valued degrees, and therefore with interval fuzzy connectives, is to deal with approximations of exact but incomplete knowledge of truth degrees provided by experts. Since the interval constructor is closed on the bounded lattices, the t-norm notion on bounded lattice is sufficient, however, here we see how to transform an arbitrary t-norm on a bounded lattice into a t-norm on its interval bounded lattice.

Proposition 4.9: Let T be a t-norm on the bounded lattice $\mathbf{L}$. Then $\mathbb{I}[T] : IL^2 \longrightarrow IL$ defined by

$$\mathbb{I}[T](X, Y) = [T(\underline{x}, \underline{y}), T(\overline{x}, \overline{y})]$$

is a t-norm on the bounded lattice $\mathbb{I}\mathbf{L}$.

Proof: Commutativity, monotonicity and neutral element ($[1, 1]$) properties of $\mathbb{I}[T]$ follow straightforward from the same properties of T. The associativity requests a bit of attention.

$$\begin{aligned} \mathbb{I}[T](X, \mathbb{I}[T](Y, Z)) &= \mathbb{I}[T](X, [T(\underline{y}, \underline{z}), T(\overline{y}, \overline{z})] \\ &= [T(\underline{x}, T(\underline{y}, \underline{z})), T(\overline{x}, T(\overline{y}, \overline{z}))] \\ &= [T(T(\underline{x}, \underline{y}), \underline{z}), T(T(\overline{x}, \overline{y}), \overline{z})] \\ &= \mathbb{I}[T]([T(\underline{x}, \underline{y}), T(\overline{x}, \overline{y})], Z) \\ &= \mathbb{I}[T](\mathbb{I}[T](X, Y), Z). \end{aligned}$$

■

Proposition 4.10: Let T be a t-norm on the bounded lattice $\mathbf{L}$. Then the projections $l : IL \longrightarrow L$ and $r : IL \longrightarrow M$ defined by

$$l(X) = \underline{x} \text{ and } r(X) = \overline{x}$$

are t-norm morphisms from $\mathbb{I}[T]$ into T.

Proof: $l(\mathbb{I}[T](X, Y)) = \underline{\mathbb{I}[T](X, Y)} = \underline{[T(\underline{x}, \underline{y}), T(\overline{x}, \overline{y})]} = T(\underline{x}, \underline{y}) = T(l(X), l(Y))$

So, l and, by analogy, r are t-norm morphisms. ■

Proposition 4.11: Let T_1 and T_2 be t-norms on the bounded lattices $\mathbf{L}$ and $\mathbf{M}$, respectively. If ρ is a t-norm morphism from T_1 into T_2, then there exists a unique t-norm morphism $\mathbb{I}[\rho]$ from $\mathbb{I}[T_1]$ into $\mathbb{I}[T_2]$ such that the following diagram commutes:

$$\begin{tikzcd} T_1 \arrow[d, "\rho"'] & \mathbb{I}[T_1] \arrow[l, "l"'] \arrow[r, "r"] \arrow[d, dashed, "\mathbb{I}[\rho]"'] & T_1 \arrow[d, "\rho"'] \\ T_2 & \mathbb{I}[T_2] \arrow[l, "l"'] \arrow[r, "r"] & T_2 \end{tikzcd}$$

Proof: Let $\mathbb{I}[\rho] : IL_1 \longrightarrow IL_2$ defined by

$$\mathbb{I}[\rho](X) = [\rho(l(X)), \rho(r(X))].$$

Since,

$\mathbb{I}[\rho](\mathbb{I}[T_1](X, Y)) = [\rho(l(\mathbb{I}[T_1](X, Y))), \rho(r(\mathbb{I}[T_1](X, Y)))] = [\rho(l([T_1(\underline{x}, \underline{y}), T_1(\overline{x}, \overline{y})]), \rho(r([T_1(\underline{x}, \underline{y}), T_1(\overline{x}, \overline{y})])] = [\rho(T_1(\underline{x}, \underline{y})), \rho(T_1(\overline{x}, \overline{y}))] \leq [T_2(\rho(\underline{x}), \rho(\underline{y})), T_2(\rho(\overline{x}), \rho(\overline{y}))] = \mathbb{I}[T_2](\mathbb{I}[\rho](X), \mathbb{I}[\rho](Y))$. $\mathbb{I}[\rho]$ is a t-norm morphism from $\mathbb{I}[T_1]$ into $\mathbb{I}[T_2]$.

Since, $l(\mathbb{I}[\rho](X)) = l([\rho(l(X)), \rho(r(X))]) = \rho(l(X))$ and $r(\mathbb{I}[\rho](X)) = r([\rho(l(X)), \rho(r(X))]) = \rho(r(X))$, the above diagram commutes.

Now suppose that ρ' is t-norm morphism from $\mathbb{I}[T_1]$ into $\mathbb{I}[T_2]$ which commutes the diagram above. Then

$$l(\rho'(X)) = \rho(l(X)) \text{ and } r(\rho'(X)) = \rho(r(X))$$

Therefore,

$$\begin{aligned} \rho'(X) &= [l(\rho'(X)), r(\rho'(X))] \\ &= [\rho(l(X)), \rho(r(X))] \\ &= \mathbb{I}[\rho](X). \end{aligned}$$

■

So, $\mathbb{I}$ could be seen as a covariant functor from $\mathcal{T}$ into $\mathcal{T}$.

V. Final remarks

This is an introductory paper which considers a well known generalization of the t-norm notion for arbitrary bounded lattices and introduces a generalization of the automorphism notion for t-norms on arbitrary bounded lattices, named t-norm morphisms. With these two generalizations we can consider a rich category having t-norms as objects and t-norm morphism as morphism. We then prove that this category is Cartesian and for the case of its subcategory where the objects are strict t-norms, we proved that it is a Cartesian closed category. Moreover we show that the usual interval construction on lattices, is a functor on those categories.

The t-norm morphisms are usual morphisms between lattice ordered monoids (l-monoid in short) which are integral, i.e. which have the universal upper bound of the lattice as the unit element of the monoid, where the t-norm is just the monoidal operation [6], [19], [31]. Since, t-conorms are also monoids, the category of l-monoid is more general than the study in this paper. Observe that properties of a super category are not always inherited by their subcategories. For example, the category of cpos is a cartesian closed category, but the category of algebraic cpos is not cartesian closed. Moreover, the category of Scott domain (which is a subcategory of algebraic cpos) is cartesian closed [21]. Thus, a further work is to analise which other usual categorical construction and properties our category has and compare it with the properties of the l-monoid category. We also plan to extend for bounded lattices other usual notions of fuzzy theory, such as t-conorms, implications, negations, additive generators, copulas, etc. and see them as categories and relate them via natural transformations.

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