Interval Valued R-Implications and Automorphisms

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Abstract

Interval fuzzy logic is firmly integrated with principles of fuzzy logic theory and interval mathematics. The former provides a complete and inclusive mathematical model of uncertainty from which the foundations of fuzzy control have widened the scope of control theory. The latter models the uncertainty and the errors in numerical computation, leading to self-validated methods. Both areas were independently developed in the mid 1960s improving the quantitative analysis of approximations to mathematically exact values, which may not be observable, representable or computable. Interval fuzzy connectives have been described in the terms of the combination of those theories. In this work, the best interval representation is considered for the study of R-implication in fuzzy logic. Based on the best interval representation, an interval fuzzy R-implication is obtained as a canonical extension satisfying the optimality property and preserving the same properties satisfied by the fuzzy R-implication. In addition, commutative diagrams relate fuzzy R-implications to interval fuzzy R-implications. This leads to the understanding of how interval automorphisms act on interval R-implications and generate other interval fuzzy R-implications.

Keywords: Fuzzy logics, R-implications, interval representations, automorphisms.

1 Introduction

Fuzzy set theory [52] may be thought as having arisen from the need of a more complete and inclusive mathematical model of uncertainty. Interval analysis [40] arose out of a need to understand and model the uncertainty and the error in numerical computations, allowing the development of computational tools for the automatic error analysis of numerical algorithms solved in digital computers. From the interval analysis emerged computational validation as a fruitful area of research [1].

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In addition, the correctness and optimality of interval analysis have been applied in technological and scientific computations (see, e.g., [7,16,34]) to provide accuracy of calculations together with automatic and rigorous control of errors of numerical computations [32]. In this sense, interval computation is adequate to deal with the imprecision or uncertainty of input values and parameters, or caused by the rounding errors that occur during the computation [40,2,41]. Interval mathematics is another form of information theory which is related to but independent from fuzzy logic.

On the other hand, fuzzy control has broadened the scope of control theory, providing a tool for describing rules using linguistic variables. The necessity for dealing with imprecision and for making decisions under uncertainties of control systems [13,43] provides the seminal motivation for development of fuzzy set theory. Fuzzy logic has been developed as a formal deductive system with a comparative notion of truth to formalize deduction under vagueness, providing a foundation for approximate reasoning using imprecise propositions based on fuzzy set theory.

The extension of classical logic connectives to the real unit interval is fundamental for the studies of fuzzy logic and therefore is essential to the development of fuzzy systems. This extension must preserve the behaviors of the connectives at the interval endpoints (crisp values) and important properties, such as commutative and associative properties of the conjunction and disjunction, resulting in the notions of triangular norms and triangular conorms, respectively.

Fuzzy implications play an important role in fuzzy logic. However, there is no consensus among researchers which exact properties of fuzzy implications should be satisfied. In the literature, several fuzzy implication properties have already been considered and their interrelationship with the other kinds of connectives are generally presented. In this paper, we are interested in fuzzy implications associated to fuzzy connectives named R-implications, which are generated by t-norms.

Additionally, whenever intervals are considered as a particular type of fuzzy set, or interval membership degrees are used in the modeling of the uncertainty in the belief of specialists, it seems natural and interesting to deal with the interval fuzzy approach.

Among several papers connecting these areas (see, e.g., [41,14,39,18,37,23]), we adopted Bedregal and Takahashi's work [9,10], where interval extensions for the fuzzy connectives, considering both correctness (accuracy) and optimality aspects, were provided [46].

The aim of this work is to introduce the concept of interval-valued R-implication (an interval generalization for R-implications) and to show that the action of the interval-valued automorphisms introduced in [23,24] preserve the interval-valued R-implications. A method for obtaining an interval-valued R-implication from an R-implication canonically, via the interval constructor, such that the resulting interval implication is the best interval representation of the R-implication, is also presented. We prove that there is a commutativity between the process for obtaining R-implications from t-norms and the process for obtaining interval-valued R-implications from interval-valued t-norms and those canonical interval constructions. We also show that the use of automorphism over R-implications, and of interval-valued automorphisms over interval-valued R-implications also comutes when the interval constructor is applied.

Observe that R-implication is a very important concepts in the context of fuzzy logic, since R-implications based on left continuous t-norms are used in the modelling of fuzzy rules that satisfy natural properties of implications (see, e.g., properties **I1** – **I10** in Sect. 4). Moreover, it takes to the residuation property and to an important family of fuzzy logics [27]. On the other hand, the use of automorphisms allows to change implication preserving its fundamental properties, and then preserving its axiomatic.

The paper is organized as follows. In Sect. 2, we discuss the conditions under which best interval representations of real functions are obtained, and present the related definitions and results. Based on these considerations, we focus attention on the interval extensions of fuzzy t-norm in Sect. 3. Further analysis of the properties satisfied by fuzzy R-implications is carried out in Sect. 4. We show that minimal properties of fuzzy implications may be extended from interval fuzzy degrees, in a natural way. In addition, a commutative diagram relating fuzzy R-implications to interval-valued fuzzy R-implications is also discussed. The action of an interval-valued automorphism on an interval-valued R-implication is analyzed in Sect. 5. The canonical construction of an interval-valued automorphism from an automorphism, including its best interval representation, and the relation between interval-valued implications and automorphism are also discussed. In Sect. 6, we conclude with the main results of this paper and some final remarks.

2 Interval Representations

Consider the real unit interval $U = [0,1] \subseteq \Re$ and let $\mathbb{U}$ be the set of subintervals of U, that is, $\mathbb{U} = \{[a,b] \mid 0 \leq a \leq b \leq 1\}$.

The interval set has two projections $l, r : \mathbb{U} \rightarrow U$ defined by $l([a,b]) = a$ and $r([a,b]) = b$, respectively. For $X \in \mathbb{U}$, $l(X)$ and $r(X)$ are also denoted by $\underline{X}$ and $\overline{X}$, respectively.

Several natural partial orders may be defined on $\mathbb{U}$ [12]. The most used orders in the context of interval mathematics are the following:

(i) *Product*: $X \leq Y$ if and only if $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$;

(ii) *Inclusion*: $X \subseteq Y$ if and only if $\underline{X} \geq \underline{Y}$ and $\overline{X} \leq \overline{Y}$.

An interval $X \in \mathbb{U}$ is said to be an interval representation of a real number α if $\alpha \in X$. Considering two interval representations X and Y of a real number α, X is said a better representation of α than Y if $X \subseteq Y$. This notion can be easily extended for tuples of n intervals $(\overrightarrow{X}) = (X_1, \ldots, X_n)$.

Definition 2.1 A function $F : \mathbb{U}^n \longrightarrow \mathbb{U}$ is an *interval representation* of a function $f : U^n \longrightarrow U$ if, for each $\overrightarrow{X} \in \mathbb{U}^n$ and $\overrightarrow{x} \in \overrightarrow{X}$, $f(\overrightarrow{x}) \in F(\overrightarrow{X})$ [46].

Definition 2.2 Let $F : \mathbb{U}^n \longrightarrow \mathbb{U}$ and $G : \mathbb{U}^n \longrightarrow \mathbb{U}$ be two interval representations of the function $f : U \longrightarrow U$. F is a *better interval representation* of f than G, denoted by $G \sqsubseteq F$, if, for each $\overrightarrow{X} \in \mathbb{U}^n$, the inclusion $F(\overrightarrow{X}) \subseteq G(\overrightarrow{X})$ holds.

2.1 The Best Interval Representation

Definition 2.3 For each real function $f : U^n \longrightarrow U$, the interval function $\widehat{f} : \mathbb{U}^n \longrightarrow \mathbb{U}$ defined by

$$\widehat{f}(\overrightarrow{X}) = \left[\inf\{f(\overrightarrow{x}) \mid \overrightarrow{x} \in \overrightarrow{X}\}, \sup\{f(\overrightarrow{x}) \mid \overrightarrow{x} \in \overrightarrow{X}\}\right] \tag{1}$$

is called *the best interval representation of* f [46].

The interval function $\widehat{f}$ is well defined and for any other interval representation F of f, $F \sqsubseteq \widehat{f}$. The interval function $\widehat{f}$ returns an interval that is narrower (has a lesser diameter) than any other interval representation of f. Thus, $\widehat{f}$ presents the *optimality property* of interval algorithms mentioned by Hickey et al. [32], when it is seen as an algorithm to compute a real function f.

Notice that if f is continuous in the usual sense, then for each $\overrightarrow{X} \in \mathbb{U}^n$, the interval function $\widehat{f}$ applied to $\overrightarrow{X}$ coincides with the image of f when applied to $\overrightarrow{X}$, that is, $\widehat{f}(\overrightarrow{X}) = f(\overrightarrow{X})$, where $f(\overrightarrow{X}) = \{f(\overrightarrow{x}) \mid \overrightarrow{x} \in \overrightarrow{X}\}$.

There are several possible notions of continuity for interval functions (see, e.g., [3,46]). In this paper we will take in consideration the Moore and Scott continuities. Another approach based on Coherence Spaces can be found in [15,17].

The main result in [46] can be adapted to our context, considering U^n instead of $\Re$, as shown in the following:

Theorem 2.4 *Let* $f : U^n \longrightarrow U$ *be a function. The following statements are equivalent:*

(i) f *is continuous;*

(ii) $\widehat{f}$ *is Scott continuous;*

(iii) $\widehat{f}$ *is Moore continuous.*

3 Interval t-norms

Considering the interval generalization proposed in [9], an interval triangular norm (*t-norm*, for short) may be considered as an interval representation of a t-norm. This generalization fits with the fuzzy principle, which means that the interval membership degree may be thought as an approximation of the exact degree.

Notice that a t-norm is a function $T : U^2 \to U$ that is commutative, associative, monotonic and has 1 as neutral element. In the following definition, a natural extension of the t-norm notion for $\mathbb{I}$ is considered, following the same approach introduced in [9].

Definition 3.1 A function $\mathbb{T} : \mathbb{U}^2 \to \mathbb{U}$ is an *interval t-norm* if it is commutative, associative, monotonic with respect to the product and inclusion order and $[1, 1]$ is a neutral element.

For the proofs of the next three propositions in this section, see [10]. The following result shows how an interval t-norm can be constructed from two given t-norms.

Proposition 3.2 *A function* $\mathbb{T} : \mathbb{U}^2 \to \mathbb{U}$ *is an interval t-norm if and only if there exist t-norms* T_1 *and* T_2 *such that* $T_1 \leq T_2$ *and* $\mathbb{T} = I[T_1, T_2]$*, where*

$$I[T_1, T_2](X, Y) = [T_1(\underline{X}, \underline{Y}), T_2(\overline{X}, \overline{Y})]. \tag{2}$$

The following proposition states that the best interval representation of a t-norm is an interval t-norm.

Proposition 3.3 *If* T *is a t-norm then* $\widehat{T} : \mathbb{U}^2 \to \mathbb{U}$ *is an interval t-norm.*

The next proposition shows that the interval representation of a t-norm coincides with the interval construction provided in Prop. 3.2 when both t-norms are the same.

Proposition 3.4 *Let T be a t-norm and $\widehat{T} : \mathbb{U}^2 \rightarrow \mathbb{U}$ be an interval t-norm. Then*

$$\widehat{T}(X, Y) = [T(\underline{X}, \underline{Y}), T(\overline{X}, \overline{Y})]. \tag{3}$$

4 Fuzzy Implication

Several definitions for fuzzy implication together with related properties have been given (see, e.g., [4,6,11,20,19,33,38,45,49,50,51]). The unique consensus in these definitions is that the fuzzy implication should present the same behavior of the classical implication for the crisp case. Thus, a binary function $I : U^2 \longrightarrow U$ is a *fuzzy implication* if it satisfies the minimal boundary conditions:

$$I(1, 1) = I(0, 1) = I(0, 0) = 1 \text{ and } I(1, 0) = 0.$$

Several reasonable properties may be required for fuzzy implications. The properties considered in this paper are listed below:

I1 : If $y \leq z$ then $I(x, y) \leq I(x, z)$;

I2 : $I(x, I(y, z)) = I(y, I(x, z))$ (exchange principle);

I3 : $I(x, y) = 1$ if and only if $x \leq y$;

I4 : $\lim_{n \to \infty} I(x, y_n) = I(x, \lim_{n \to \infty} y_n)$ (right-continuity);

I5 : If $x \leq z$ then $I(x, y) \geq I(z, y)$;

I6 : $I(x, 1) = 1$;

I7 : $I(0, x) = 1$ (dominance of falsity);

I8 : $I(1, x) = x$ (neutrality of truth);

I9 : $I(x, y) \geq y$;

I10 : $I(x, x) = 1$ (identity);

The next proposition presents a relation among the properties **I1–10**, showing that the four first are strongest than the others.

Proposition 4.1 *Let I be a fuzzy implication satisfying* **I1**, **I2** *and* **I3**. *Then I also satisfies* **I5** – **I10**.

Proof. See [11, Lemma 1 (xi)]. □

4.1 R-implications

Let T be a t-norm. Then the equation

$$I_T(x, y) = \sup\{z \in [0, 1] \mid T(x, z) \leq y\}, \forall x, y \in [0, 1] \tag{4}$$

defines a fuzzy implication, called *R-implication* or *residuum* of T [20,19,36,11,4,22,47]. The R-implication arises from the notion of residuum in Intuitionistic Logic [5] or, equivalently, from the notion of residue in the theory of lattice-ordered semigroups [21]. Observe that the R-implication is well-defined only if the t-norm is left-continuous [5] [25,4,47]. This justifies the name "residuum of T", since the R-implication satisfies the residuation condition when the underlying t-norm is left continuous:

$$T(x, z) \leq y \text{ if and only if } I_T(x, y) \geq z. \tag{5}$$

Moreover, a t-norm T is left-continuous if and only if it satisfies the residuation condition [22].

The main results relating the R-implication and the properties **I1** – **I10** are presented in the following.

Theorem 4.2 *Let $I : U^2 \to U$ be a fuzzy implication. Then, I is an R-implication with a left-continuous underlying t-norm if and only if I satisfies the properties* **I1** *to* **I4**.

Proof. See [48,19,4,47]. □

Since, by Prop. 4.1, **I5** – **I10** follow directly from **I1** – **I3**, then R-implications with left-continuous underlying t-norms satisfy the properties **I1** – **I10**.

5 Interval-valued Fuzzy Implications

According to the idea that values in interval mathematics are identified with degenerate intervals, the minimal properties of fuzzy implications can be naturally extended from interval fuzzy degrees, whenever the respective degenerate intervals are considered. Thus, a function $\mathbb{I} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is an *interval fuzzy implication* if the following conditions hold:

$$\mathbb{I}([1,1],[1,1]) = \mathbb{I}([0,0],[0,0]) = \mathbb{I}([0,0],[1,1]) = [1,1]; \tag{6}$$

$$\mathbb{I}([1,1],[0,0]) = [0,0]. \tag{7}$$

Some extra properties can be naturally extended:

$\mathbb{I}$1 : If $Y \leq Z$ then $\mathbb{I}(X, Y) \leq \mathbb{I}(X, Z)$,

$\mathbb{I}$2 : $\mathbb{I}(X, \mathbb{I}(Y, Z)) = \mathbb{I}(Y, \mathbb{I}(X, Z))$,

$\mathbb{I}$3 : $\mathbb{I}(X, Y) = [1, 1]$ if and only if $\overline{X} \leq \underline{Y}$,

$\mathbb{I}$4a : $\mathbb{I}_Y(X) = \mathbb{I}(X, Y)$ is Moore-continuous,

$\mathbb{I}$4b : $\mathbb{I}_Y(X) = \mathbb{I}(X, Y)$ is Scott-continuous,

$\mathbb{I}$5 : If $X \leq Z$ then $\mathbb{I}(X, Y) \geq \mathbb{I}(Z, Y)$,

$\mathbb{I}$6 : $\mathbb{I}([0, 0], X) = [1, 1]$,

$\mathbb{I}$7 : $\mathbb{I}(X, [1, 1]) = [1, 1]$,

$\mathbb{I}$8 : $\mathbb{I}([1, 1], X) = X$,

$\mathbb{I}$9 : $\mathbb{I}(X, Y) \geq Y$,

$\mathbb{I}$10 : $\overline{\mathbb{I}(X, X)} = 1$

Observe that it is always possible to obtain an interval fuzzy implication from any fuzzy implication canonically. The interval fuzzy implication also meets the optimality

[5] A t-norm T is said to be left-continuous whenever $\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y)$. [36,22]

property and preserves the same properties satisfied by the fuzzy implication. In the following two propositions, the best interval representation of a fuzzy implication is shown as an inclusion-monotonic function in both arguments. The related proofs are straightforward, following from the definition of $\widehat{I}$ as a particular case of the equation (1).

Proposition 5.1 *If I is a fuzzy implication then $\widehat{I}$ is an interval fuzzy implication.*

Proof. See [10]. □

Proposition 5.2 *Let I be a fuzzy implication. Then, for each $X_1, X_2, Y_1, Y_2 \in \mathbb{U}$, if $X_1 \subseteq X_2$ and $Y_1 \subseteq Y_2$ then it holds that $\widehat{I}(X_1, Y_1) \subseteq \widehat{I}(X_2, Y_2)$.*

Proof. It is straightforward. □

The next theorem states that the best interval representation of a fuzzy implication preserves, in some sense, the properties **I1–I10** listed in Sect. 4.

Theorem 5.3 *Let I be a fuzzy implication. If I satisfies a property* **Ik**, *for $k = 1, \ldots, 10$, then $\widehat{I}$ satisfies the property* $\mathbb{I}$**k**.

Proof.

$\mathbb{I}$**1**: If I satisfies **I1**, then it holds that $\widehat{I}(X, Y) = [\inf\{I(x, \underline{Y})|x \in X\}, \sup\{I(x, \overline{Y})|x \in X\}]$ and $\widehat{I}(X, Z) = [\inf\{I(x, \underline{Z})|x \in X\}, \sup\{I(x, \overline{Z})|x \in X\}]$. Thus, if $Y \leq Z$ then, for each $x \in X$, it is valid that $I(x, \underline{Y}) \leq I(x, \underline{Z})$ and $I(x, \overline{Y}) \leq I(x, \overline{Z})$. It follows that $\inf\{I(x, \underline{Y})|x \in X\} \leq \inf\{I(x, \underline{Z}) \mid x \in X\}$ and $\sup\{I(x, \overline{Y})|x \in X\} \leq \sup\{I(x, \overline{Z})|x \in X\}$. Therefore, we conclude that $\widehat{I}(X, Y) \leq \widehat{I}(X, Z)$.

$\mathbb{I}$**2**: First, observe that, if $u \in \widehat{I}(X, Y)$, then there exist $x \in X$ and $y \in Y$ such that $I(x, y) = u$. Thus, if $u \in \widehat{I}(X, \widehat{I}(Y, Z))$, then there exist $x \in X$, $y \in Y$ and $z \in Z$ such that $I(x, I(y, z)) = u$. But, by **I2**, one has that $u = I(y, I(x, z))$. It follows that $u \in \widehat{I}(Y, \widehat{I}(X, Z))$ and, therefore, it holds that $\widehat{I}(X, \widehat{I}(Y, Z)) \subseteq \widehat{I}(Y, \widehat{I}(X, Z))$. Analogously, if $u \in \widehat{I}(Y, \widehat{I}(X, Z))$, then there exist $x \in X$, $y \in Y$ and $z \in Z$ such that $I(y, I(x, z)) = u$. However, by **I2**, one has that $u = I(x, I(y, z))$. It follows that $u \in \widehat{I}(X, \widehat{I}(Y, Z))$ and, therefore, it results that $\widehat{I}(Y, \widehat{I}(X, Z)) \subseteq \widehat{I}(X, \widehat{I}(Y, Z))$. Hence, we conclude that $\widehat{I}(X, \widehat{I}(Y, Z)) = \widehat{I}(Y, \widehat{I}(X, Z))$.

$\mathbb{I}$**3**: One has that $\widehat{I}(X, Y) = [1, 1]$ if and only if $\inf\{I(x, y)|x \in X, y \in Y\} = 1 = \sup\{I(x, y)|x \in X, y \in Y\}$. However, this is only possible if and only if $\{I(x, y)|x \in X, y \in Y\} = \{1\}$, and, therefore, if and only if $I(x, y) = 1$, for each $x \in X$ and $y \in Y$. Hence, since I satisfies **I3**, this happens if and only if, for each $x \in X$ and $y \in Y$, it is valid that $x \leq y$. This is only possible if and only if $\overline{X} \leq \underline{Y}$.

$\mathbb{I}$**4**: For each $x \in X$, let $I_x : U \longrightarrow U$ be defined by $I_x(y) = I(x, y)$. Thus, I is right-continuous if and only if, for each $x \in X$, I_x is continuous. If I_x is continuous then, by Theorem 2.4, $\widehat{I_x}$ is Scott (Moore) continuous. It follows that

$$
\begin{aligned}
\widehat{I}(X, Y) &= [\inf\{I(x, y)|x \in X \text{ and } y \in Y\}, \sup\{I(x, y)|x \in X \text{ and } y \in Y\}] \\
&= [\inf\{\inf\{I(x, y)|y \in Y\} \mid x \in X\}, \sup\{\sup\{I(x, y)|y \in Y\}|x \in X\}] \\
&= \textstyle\bigcup_{x \in X} [\inf\{I(x, y)|y \in Y\}, \sup\{I(x, y)|y \in Y\}] \\
&= \textstyle\bigcup_{x \in X} \widehat{I_x}(Y)
\end{aligned}
$$

and then, considering that I_x is (topologically) continuous and the union preserves continuity, one concludes that $\widehat{I}$ is also Scott (Moore) continuous.

$\mathbb{I}$**5**: If I satisfies **I5**, then it holds that $\widehat{I}(X,Y) = [\inf\{I(\overline{X},y)|y \in Y\}, \sup\{I(\underline{X},y)|y \in Y\}]$ and $\widehat{I}(Z,Y) = [\inf\{I(\overline{Z},y)|y \in Y\}, \sup\{I(\underline{Z},y)|y \in Y\}]$. Thus, if $X \leq Z$ then, for each $y \in Y$, it is valid that $I(\underline{X},y) \leq I(\underline{Z},y)$ and $I(\overline{X},y) \leq I(\overline{Z},y)$. It follows that $\inf\{I(\overline{Z},y)|y \in Y\} \leq \inf\{I(\overline{X},y) \mid y \in Y\}$ and $\sup\{I(\underline{Z},y)|y \in Y\} \leq \inf\{I(\underline{X},y)|y \in Y\}$. Therefore, we conclude that $\widehat{I}(X,Y) \geq \widehat{I}(Z,Y)$.

$\mathbb{I}$**6**: Observe that $\widehat{I}([0,0],X) = [\inf\{I(0,x)|x \in X\}, \sup\{I(0,x)|x \in X\}]$. Since I satisfies **I6**, it follows that $I(0,x) = 1$, for each $x \in X$. Then, it holds that $\{I(0,x)|x \in X\} = \{1\}$, and, therefore, it is valid that $\widehat{I}([0,0],X) = [1,1]$.

$\mathbb{I}$**7**: Observe that $\widehat{I}(X,[1,1]) = [\inf\{I(x,1)|x \in X\}, \sup\{I(x,1)|x \in X\}]$. Since I satisfies **I7**, it follows that $I(x,1) = 1$, for each $x \in X$. Then, it holds that $\{I(x,1)|x \in X\} = \{1\}$, and, therefore, it is valid that $\widehat{I}(X,[1,1]) = [1,1]$.

$\mathbb{I}$**8**: Observe that $\widehat{I}([1,1],X) = [\inf\{I(1,x)|x \in X\}, \sup\{I(1,x)|x \in X\}]$. Since I satisfies **I8**, it follows that $I(1,x) = x$, for each $x \in X$. Then, it holds that $\{I(1,x)|x \in X\} = X$, and, therefore, it is valid that $\widehat{I}([1,1],X) = X$.

$\mathbb{I}$**9**: One has that $\widehat{I}(X,Y) = [\inf\{I(x,y)|x \in X, y \in Y\}, \sup\{I(x,y)|x \in X, y \in Y\}]$. Since I satisfies **I9**, it is valid that $I(x,y) \geq y$, for each $x \in X$ and $y \in Y$. Therefore, it holds that $\inf\{I(x,y)|x \in X, y \in Y\} \geq \underline{Y}$ and $\sup\{I(x,y)|x \in X, y \in Y\} \geq \overline{Y}$. So, one concludes that $\widehat{I}(X,Y) \geq Y$.

$\mathbb{I}$**10**: Since I satisfies **I10**, it holds that $\{I(x,x)|x \in X\} = \{1\}$. Thus, since $\{I(x,x)|x \in X\} \subseteq \{I(x,y)|x \in X, y \in X\}$, it follows that $\overline{\widehat{I}(X,X)} = 1$.

□

The next corollary indicates that the best interval representation of a fuzzy implication satisfying **I1**–**I3** satisfies the properties $\mathbb{I}$**1**–$\mathbb{I}$**3** and $\mathbb{I}$**5**–$\mathbb{I}$**10** listed in the beginning of this section.

Corollary 5.4 *Let $I : U^2 \longrightarrow U$ be a fuzzy implication satisfying* **I1**, **I2** *and* **I3**. *Then $\widehat{I}$ satisfies* $\mathbb{I}$***1***–$\mathbb{I}$***3*** *and* $\mathbb{I}$***5***–$\mathbb{I}$***10***.

Proof. It is straightforward, following from Prop. 4.1 and Theorem 5.3. □

The next proposition provides, for the best interval representation of a fuzzy implication satisfying the properties **I1**, **I2** and **I3**, a more concrete and simpler characterization of the endpoints than the one given by Equation (1).

Proposition 5.5 *Let $I : U^2 \longrightarrow U$ be a fuzzy implication satisfying the properties* **I1**, **I2** *and* **I3**. *Then a characterization of $\widehat{I}$ can be obtained as*

$$\widehat{I}(X,Y) = [I(\overline{X},\underline{Y}), I(\underline{X},\overline{Y})]. \tag{8}$$

Proof. By Prop. 4.1, I also satisfies **I5** and, therefore, it holds that $\widehat{I}(X,Y) = [I(\overline{X},\underline{Y}), I(\underline{X},\overline{Y})]$ (see also [10]). □

Similar characterizations can be obtained for other cases. For example, if I satisfies the properties **I1**, **I5** and **I10**, then it holds that that $\widehat{I}(X,X) = [I(\overline{X},\underline{X}),1]$.

5.1 Interval-valued R-implications

Definition 5.6 An interval fuzzy implication $\mathbb{I}$ is an *interval R-implication* if there is an interval t-norm $\mathbb{T}$ such that $\mathbb{I} = \mathbb{I}_{\mathbb{T}}$, where

$$\mathbb{I}_{\mathbb{T}}(X,Y) = \sup\{Z \in \mathbb{U} | \mathbb{T}(X,Z) \leq Y\}. \tag{9}$$

Observe that, in Equation (9), the supremum is determined considering the product order, and, therefore, it results from the supremum considering the usual order on the real numbers (the interval endpoints).

The next proposition shows that, analogously to R-implications, interval R-implications satisfy the properties $\mathbb{I}$**1**, $\mathbb{I}$**2** e $\mathbb{I}$**3**.

Proposition 5.7 *Let $\mathbb{I}$ be an interval fuzzy implication. If $\mathbb{I}$ is an interval R-implication then $\mathbb{I}$ satisfies $\mathbb{I}$**1**, $\mathbb{I}$**2** and $\mathbb{I}$**3**.*

Proof. Let $\mathbb{T}$ be the underlying interval t-norm of $\mathbb{I}$, that is, $\mathbb{I} = \mathbb{I}_{\mathbb{T}}$. It follows that:

$\mathbb{I}$**1**: If $Y \leq Z$ and $\mathbb{T}(X,Z') \leq Y$ then it holds that $\mathbb{T}(X,Z') \leq Z$. It follows that $\{Z' \in \mathbb{U} \mid \mathbb{T}(X,Z') \leq Y\} \subseteq \{Z' \in \mathbb{U} \mid \mathbb{T}(X,Z') \leq Z\}$, and, therefore, it is valid that $\mathbb{I}_{\mathbb{T}}(X,Y) \leq \mathbb{I}_{\mathbb{T}}(X,Z)$.

$\mathbb{I}$**2**: Let T_1 and T_2 be the t-norms related to $\mathbb{T}$ by Prop. 3.2. Then it follows that:

$$\begin{aligned}
&\mathbb{I}_{\mathbb{T}}(X, \mathbb{I}_{\mathbb{T}}(Y,Z)) \\
&= \sup\{X' \in \mathbb{U} \mid \mathbb{T}(X,X') \leq \sup\{Y' \in \mathbb{U} | \mathbb{T}(Y,Y') \leq Z\}\} && \text{by Eq. (9)} \\
&= \sup\{X' \in \mathbb{U} | \mathbb{T}(X,X') \leq \sup\{Y' \in \mathbb{U} | [T_1(\underline{Y},\underline{Y'}), T_2(\overline{Y},\overline{Y'})] \leq Z\}\} && \text{by Eq. (2)} \\
&= \sup\{X' \in \mathbb{U} | \mathbb{T}(X,X') \leq \\
&\qquad [\sup\{\underline{Y'} \in U \mid T_1(\underline{Y},\underline{Y'}) \leq \underline{Z}\}, \sup\{\overline{Y'} \in U | T_2(\overline{Y},\overline{Y'}) \leq \overline{Z}\}\} \\
&= \sup\{X' \in \mathbb{U} | \mathbb{T}(X,X') \leq [I_{T_1}(\underline{Y},\underline{Z}), I_{T_2}(\overline{Y},\overline{Z})]\} && \text{by Eq. (4)} \\
&= [\sup\{\underline{X'} \in U \mid T_1(\underline{X},\underline{X'}) \leq I_{T_1}(\underline{Y},\underline{Z})\}, \\
&\qquad \sup\{\overline{X'} \in U | T_2(\overline{X},\overline{X'}) \leq I_{T_2}(\overline{Y},\overline{Z})\}] && \text{by Eq. (2)} \\
&= [I_{T_1}(\underline{X}, I_{T_1}(\underline{Y},\underline{Z})), I_{T_2}(\overline{X}, I_{T_2}(\overline{Y},\overline{Z}))] && \text{by Eq. (4)} \\
&= [I_{T_1}(\underline{Y}, I_{T_1}(\underline{X},\underline{Z})), I_{T_2}(\overline{Y}, I_{T_2}(\overline{X},\overline{Z}))] && \text{by Theorem 4.2} \\
&= \mathbb{I}_{\mathbb{T}}(Y, \mathbb{I}_{\mathbb{T}}(X,Z)), \text{ by the inverse construction.}
\end{aligned}$$

$\mathbb{I}$**3**: Observe that $\mathbb{I}_{\mathbb{T}}(X,Y) = [1,1]$ if and only if $\sup\{Z \in \mathbb{U} \mid \mathbb{T}(X,Z) \leq Y\} = [1,1]$ if and only if $\{Z \in \mathbb{U} \mid \mathbb{T}(X,Z) \leq Y\} = \mathbb{U}$ if and only if $X = \mathbb{T}(X,[1,1]) \leq Y$.

□

The next theorem states that the best interval representation of an R-implication that is obtained from a left continuous t-norm coincides with the interval R-implication obtained from the best interval representation of the t-norm. Then, this theorem provides a simpler characterization for the best interval representation of an R-implication than the one obtained by the direct application of Equation (1).

Theorem 5.8 *Let T be a left continuous t-norm. Then it holds that*

$$\widehat{I_T} = \mathbb{I}_{\widehat{T}}. \tag{10}$$

Proof. It follows that:

$$
\begin{aligned}
&\mathbb{I}_{\widehat{T}}(X,Y) \\
&= \sup\{Z \in \mathbb{U} | \widehat{T}(X,Z) \leq Y\} && \text{by Eq. (9)} \\
&= \sup\{[\underline{Z},\overline{Z}] \in \mathbb{U} | [T(\underline{X},\underline{Z}), T(\overline{X},\overline{Z})] \leq [\underline{Y},\overline{Y}]\} && \text{by Eq. (3)} \\
&= \sup\{[\underline{Z},\overline{Z}] \in \mathbb{U} | T(\underline{X},\underline{Z}) \leq \underline{Y} \wedge T(\overline{X},\overline{Z}) \leq \overline{Y}\} && \text{by the Def. of the product order} \\
&= [\sup\{\underline{Z} \in U | T(\underline{X},\underline{Z}) \leq \underline{Y}\}, \sup\{\overline{Z} \in U | T(\overline{X},\overline{Z}) \leq \overline{Y}\}] && \text{by the Def. of supremum} \\
&= [I_T(\underline{X},\underline{Y}), I_T(\overline{X},\overline{Y})] && \text{by Eq. (4)} \\
&\subseteq [I_T(\overline{X},\underline{Y}), I_T(\underline{X},\overline{Y})] && \text{by Theorem (4.2)} \\
&= [\inf\{I_T(x,y) | x \in X \wedge y \in Y\}, \sup\{I_T(x,y) | x \in X \wedge y \in Y\}] && \text{by Theorem (4.2)} \\
&= \widehat{I_T}(X,Y) && \text{by Eq. (1)}
\end{aligned}
$$

On the other hand, whenever $x \in X$ and $y \in Y$, then, for each $z \in U$, if $T(x,z) \leq y$ then, by left continuity of T, there exists $Z \in \mathbb{U}$ such that $\widehat{T}(X,Z) \leq Y$. It follows that $I_T(x,y) = \sup\{z \in U | T(x,z) \leq y\} \in \sup\{Z \in \mathbb{U} | \widehat{T}(X,Z) \leq Y\} = \mathbb{I}_{\widehat{T}}(X,Y)$. Therefore, $\mathbb{I}_{\widehat{T}}$ is an interval representation of I_T, but $\widehat{I_T}$ is the best representation of I_T. So, for each $X \in \mathbb{U}$ and $Y \in \mathbb{U}$, it holds that $\widehat{I_T}(X,Y) \subseteq \mathbb{I}_{\widehat{T}}(X,Y)$. □

The next corollary showing that the best interval representation of an R-implication is an interval R-implication follows directly.

Corollary 5.9 *If I is an R-implication then $\widehat{I}$ is an interval R-implication.*

The above results together with Theorem 5.8 state the commutativity of the diagram presented in Fig. 1, where $\mathcal{C}(T)$ ($\mathcal{C}(\mathbb{T})$) denotes the class of (interval) t-norms and $\mathcal{C}(I)$ ($\mathcal{C}(\mathbb{I})$) is the class of (interval) R-implications.

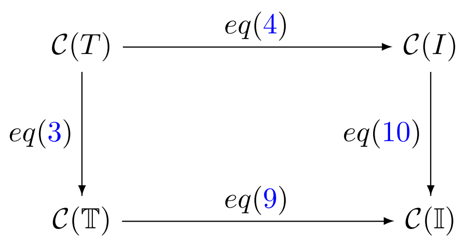

Fig. 1. Commutative diagram relating R-implications with interval R-implications

6 Interval-valued Automorphisms

Definition 6.1 A mapping $\rho : U \longrightarrow U$ is an *automorphism* if it is bijective and monotonic (i.e., $x \leq y$ implies that $\rho(x) \leq \rho(y)$) [35,42]. $Aut(U)$ denotes the set of automorphisms.

An equivalent definition, given in [11], says that $\rho : U \longrightarrow U$ is an automorphism if it is a continuous and strictly increasing function such that $\rho(0) = 0$ and $\rho(1) = 1$.

Automorphisms are closed under composition, that is, if ρ and ρ' are automorphisms then $\rho \circ \rho'(x) = \rho(\rho'(x))$ is also an automorphism. In addition, the inverse of an automorphism is also an automorphism.

Let ρ be an automorphism and I be a fuzzy implication. The *action* of ρ on I, denoted by I^ρ, defined as

$$I^\rho(x,y) = \rho^{-1}(I(\rho(x),\rho(y))), \tag{11}$$

is a fuzzy implication. Moreover, if I is an R-implication then I^ρ is also an R-implication.

6.1 Canonical Construction of an Interval Automorphism

A mapping $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an *interval automorphism* if it is bijective and monotonic with respect to the product order [23,24] (i.e., $X \leq Y$ implies that $\varrho(X) \leq \varrho(Y)$). The set of all interval automorphisms $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is denoted by $Aut(\mathbb{U})$.

The next theorem shows that each interval automorphism can be constructed from an automorphism.

Theorem 6.2 *Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be an interval automorphism. Then there exists an automorphism $\rho : U \longrightarrow U$ such that*

$$\varrho(X) = [\rho(\underline{X}), \rho(\overline{X})]. \tag{12}$$

Proof. See Theorem 2 of [23]. □

The Equation (12) also provides a canonical construction of interval automorphisms from automorphisms and, therefore, a bijection between the sets $Aut(U)$ and $Aut(\mathbb{U})$ (Theorem 3 of [23]).

6.2 The Best Interval Representation of an Automorphism

For the proofs of the next five propositions in this section, see [9].

In the following, we present interval automorphisms from the point of view of its representation.

Theorem 6.3 (Automorphism Representation Theorem) *Let $\rho : U \to U$ be an automorphism. Then $\widehat{\rho}$ is an interval automorphism and its characterization can be obtained as:*

$$\widehat{\rho}(X) = [\rho(\underline{X}), \rho(\overline{X})]. \tag{13}$$

Then, interval automorphisms are the best interval representations of automorphisms.

Notice that t-norms are required, by definition, to satisfy $\subseteq$-monotonicity, but nevertheless this property is not required by the definition of interval automorphisms. In the following corollary, we show that interval automorphisms also are $\subseteq$-monotonic [9].

Corollary 6.4 *If ϱ is an interval automorphism then ϱ is inclusion monotonic, that is, if $X \subseteq Y$ then $\varrho(X) \subseteq \varrho(Y)$.*

Analogously, considering the alternative definition of automorphism provided in [11], we present alternative characterizations for interval automorphisms based on the Moore's and Scott's definitions of continuity.

A function $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is *strictly increasing* if, for each $X, Y \in \mathbb{U}$, whenever $X < Y$ (i.e., $X \leq Y$ and $X \neq Y$) then $\varrho(X) < \varrho(Y)$.

Proposition 6.5 *$\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an interval automorphism if and only if ϱ is Moore-continuous, strictly increasing, $\varrho([0,0]) = [0,0]$ and $\varrho([1,1]) = [1,1]$.*

Corollary 6.6 *Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be a Moore-continuous and strictly increasing function such that $\varrho([0,0]) = [0,0]$ and $\varrho([1,1]) = [1,1]$. Then there exists an automorphism ρ such that $\varrho = \widehat{\rho}$.*

Analogous result can be obtained for the case of Scott-continuity.

The next proposition shows the action of the inverse with respect to the composition of interval automorphisms.

Proposition 6.7 *Let ϱ_1 and ϱ_2 be interval automorphisms. Then it holds that*

$$(\varrho_1 \circ \varrho_2)^{-1} = \varrho_2^{-1} \circ \varrho_1^{-1}. \tag{14}$$

The next theorem states that the best interval representation of a fuzzy implication, obtained from the action of an automorphism on a fuzzy implication, coincides with the action of the best interval representation of the automorphism on the best interval representation of the fuzzy implication. In other words, the best interval representation preserves the action of automorphisms on fuzzy implications.

Theorem 6.8 *Let I be an implication and ρ be an automorphism. Then it holds that*

$$\widehat{I^{\rho}} = \widehat{I}^{\widehat{\rho}}. \tag{15}$$

Proof. See [44]. □

6.3 Interval Automorphism Acting on Interval R-implication

In the following theorem, we show how interval automorphisms act on interval R-implications, generating new interval R-implications.

Theorem 6.9 *Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be an interval automorphism and $\mathbb{I} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ be an interval R-implication. Then the mapping $\mathbb{I}^{\varrho} : \mathbb{U}^2 \longrightarrow \mathbb{U}$ is an interval R-implication, defined by*

$$\mathbb{I}^{\varrho}_{\mathbb{T}}(X,Y) = \mathbb{I}_{\mathbb{T}^{\varrho}}(X,Y). \tag{16}$$

Proof. It follows that:

$$\begin{aligned}
\mathbb{I}^{\varrho}_{\mathbb{T}}(X,Y) &= \varrho^{-1}(\mathbb{I}_{\mathbb{T}}(\varrho(X),\varrho(Y))) && \text{by Eq. (16)}\\
&= \varrho^{-1}(sup\{\varrho(Z) \in \mathbb{U} | \mathbb{T}(\varrho(X),\varrho(X)) \leq \varrho(Y)\}) && \text{by Eq. (9)}\\
&= (sup\{(\varrho^{-1} \circ \varrho)Z \in \mathbb{U} | \mathbb{T}((\varrho^{-1} \circ \varrho)X, (\varrho^{-1} \circ \varrho)Z) \leq (\varrho^{-1} \circ \varrho)Y)\})\\
&= (sup\{\varrho(Z) \in \mathbb{U} | \mathbb{T}^{\varrho}(X,Z) \leq Y\}) && \text{by Eq. (9)}\\
&= \mathbb{I}_{\mathbb{T}^{\varrho}}(X,Y).
\end{aligned}$$

□

Observe that Theorem 6.9 states that applying an interval automorphism to an R-Implication presents the same effect than applying it to a t-norm and then obtaining an interval R-implication. Then, this theorem guarantees that whenever an interval R-implication is submitted to an interval automorphism, a new interval R-implication is then generated, which means that interval automorphisms may be applied in order to generate new interval R-implications.

According to Theorem 6.8, the commutative diagram pictured in Fig. 2 holds.

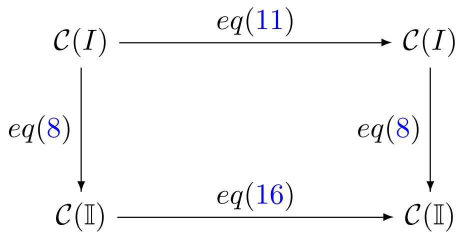

Fig. 2. Commutative diagram relating R-implications, automorphisms, interval R-implications and interval automorphisms

Based on Prop. 6.7 and Theorems 6.8 and 6.9, (interval) R-implications and (interval) automorphisms can be seen as objects and morphism, respectively, of the category $\mathfrak{C}(\mathcal{C}(I), Aut(I))$ ($\mathfrak{C}(\mathcal{C}(\mathbb{I}), Aut(\mathbb{I}))$), respectively. In a categorical approach, the action of an interval automorphism on an interval R-implication can be conceived as a covariant functor whose application over R-implications and automorphisms in $\mathfrak{C}(\mathcal{C}(I), Aut(I))$ returns the related best interval representations in $\mathfrak{C}(\mathcal{C}(\mathbb{I}), Aut(\mathbb{I}))$.

7 Final Remarks

Research on extending the ideas of validation to fuzzy logic allow researchers the capability of determining the quality of solutions within the range of uncertainty associated with the problem. The synergism between fuzzy logic and interval analysis may be used to underlie the logic system for expert systems.

We observe that the work in [29,30,31] explains that large parts of fuzzy set theory can be seen as subfields of Sheaf Theory [28], actually, of the theory of complete Ω-valued sets, showing that several key concepts of the fuzzy set theory can be naturally described in terms of subsheaves of constant sheaves and related concepts. One important claim is that fuzzy theorists are not able to give proper solutions for some problems (see, e.g., [26], and also [31] for a discussion on the quotient problem) that can be properly explained by sheaf theory. However, to be able to deal with those concepts some fundamental knowledge from sheaf theory is inevitable, as pointed out in [29].

This paper presents R-implications as the logical counter-part of the algebraic semantics for Fuzzy Set Theory, regarded to Interval-Valued varying sets. Although this concept may be defined as a quite standard sheaf construction, there is no need to know Sheaf Theory and Topoi in order to understand the presentation given here.

The results concerning interval-valued R-implications and automorphisms presented in this paper extend our previous work in [9,44,8]. Although the methodology used here is analogous to onde applied in [44,8] for interval-valued QL-implications and S-implications, we observe that, whereas S-implications, for example, are obtained directly from t-norms and fuzzy negations, R-implications are obtained as limits (supremum) of applications of t-norms, which led us to a different and more elegant approach in this presentation, and in the proofs of propositions and theorems.

Throughout this paper, intervals are used to model the uncertainty of specialists' information related to truth values in the fuzzy propositional calculus: the basic systems are based on interval t-norms, that is, using subsets of the real unit interval as the standard sets of truth degrees and applying continuous t-norms. Thus, the standard truth interval function of an R-implication can be obtained.

In addition, we mainly discussed under which conditions generalized fuzzy R-implications applied to interval values preserve properties of canonical forms generated by interval t-norms. It was shown that properties of fuzzy logic may be naturally extended for interval fuzzy degrees considering the respective degenerated intervals. The significance of an interval fuzzy R-implication was emphasized, showing that R-implications can be constructed from interval automorphisms that are preserved by the interval canonical representations.

The development of computer systems based on interval fuzzy logic would inform researchers on how to provide better information requisite for manipulating data types most accurately and efficiently. The study of interval fuzzy logic improves the use of interval methods to develop solutions to problems in the fuzzy set domain and vice versa.

These results are important not only to analyze deductive systems in mathematical depth but also to develop methods based on interval fuzzy logic. They integrate two important features: the accuracy criteria and the optimality property of interval computations, and a formal mathematical theory for the representation of uncertainty, concerned with the fuzzy set theory.

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