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# The Interval Constructor on Some Classes of ML-algebras

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**Abstract**

Monoidal logic, ML for short, which formalized the fuzzy logics of continuous t-norms and their residua, has arisen great interest, since it has been applied to fuzzy mathematics, artificial intelligence, and other areas. It is clear that fuzzy logics basically try to represent imperfect or fuzzy information aiming to model the natural human reasoning. On the other hand, in order to deal with imprecision in the computational representation of real numbers, the use of intervals have been proposed, as it can guarantee that the results of numerical computation are in a bounded interval, controlling, in this way, the numerical errors produced by successive roundings. There are several ways to connect both areas; the most usual one is to consider interval membership degrees. ML, whose algebraic counterpart is a ML-algebra, seems an interesting structure due to the fact that by adding some properties to that algebra, it is possible to reach different classes of residuated lattices. In this sense, we propose to apply an interval constructor to ML-algebras and some of their subclasses, in order to verify some properties within these algebras, in addition to the analysis of the algebraic aspects of them.

*Keywords:* Fuzzy logics, Monoidal logic, Interval mathematics, Residuated lattices, ML-algebras.

## 1 Introduction

Everyday natural language contains a lot of inexact and vague information, or in other words, fuzzy information. In order to represent and manipulate this kind of data, we need more than the classical logic, which only admits truth or falseness. The mathematical modeling of fuzzy concepts was presented by Lofti Zadeh in [42] considering that meaning, in natural language, was a matter of degree. A proposition was no longer simply true or false; instead, a real value in the interval $[0, 1]$ was considered to indicate how much that proposition was believed to be true.

This led to the development of many studies concerning fuzzy logics. In the domain of engineering and applied sciences, for instance, they have been studied as a tool to deal with the natural uncertainty of knowledge and to represent the uncertainty of human reasoning [34]. In this way, fuzzy systems try to reason in a

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similar way human beings do by attempting to transform vague information into adequate solutions to given problems or by making decisions and taking actions using this imprecise knowledge. In a different direction of study, fuzzy logics, in a certain way, enrich classical logic. They act as a symbolic logic with a comparative notion of truth, having as a base the classical logic, once the connectives must behave as in the classical logic on the extremes values. Some examples of fuzzy logics are: Monoidal Logic (ML, for short) introduced by Höhle in [24]; Monoidal t-norm based Logic (MTL) introduced by Esteva and Godo in [14]; Basic Logic, (BL, presented by Hájek in [23]); Gödel Logic (G, in [20]); and Łukasiewicz Logic (Ł, in [30]).

In systems with the approach that not only the available information is vague or uncertain, but also the membership degrees, it is common to use the interval fuzzy logics. Once the membership degrees are uncertain, they are represented by intervals in the unit interval $[0, 1]$. Interval-valued fuzzy sets were introduced independently by Zadeh [43] and others authors (e.g., [22], [26], [35]) in the 70's. The connection between the fuzzy theory and interval mathematics has been studied concerning different points of view like, for example, in [11], [12], [13], [16], [17], [28], [29], [31], [33], [32], [39] and [41]. And a good contribution for the formal study of interval fuzzy logics was done by Bedregal and Takahashi, in [7], where an interval t-norm was seen as an interval representation of t-norms in the sense of [36]. This work was extended in [5] for the bounded lattice context. Moreover, from a categorical point of view, it was proved in [1] the agreement between the interval generalization of automorphisms and this t-norm generalization, in the sense that the interval constructor of automorphisms and t-norms could be seen as a functor preserving the action of automorphisms on t-norms. Thus, it was obtained the best interval representation (in the sense of [7]) of any t-norm and automorphism leading to handle with the optimality of interval fuzzy algorithms.

Recently, fuzzy logics based on t-norms and their residua have been widely investigated. In particular, Höhle's Monoidal Logic (ML), whose algebraic counterpart is the complete class of residuated lattices (namely a ML-algebra), seems quite interesting due to the possibility of reaching other logics just by adding some axioms. Moreover, it is also possible to add properties to ML-algebras and get different classes of residuated lattices, i.e. a prelinear ML-algebra is a MTL-algebra.

In this way, our work seeks to introduce the interval constructor to ML-algebras and some of their subclasses. Besides, it verifies some properties within these algebras, in addition to the analysis of the algebraic aspects of them. Our main motivation is to show that some classes of ML-logics can be interpreted regarding fuzzy degrees, in the usual sense, as well as interval fuzzy degrees and therefore, considering some uncertainty in the expert evaluation. Thus, if for some ML-algebra $\mathcal{A}$ and a propositional formula $\varphi$, $\mathcal{A} \models_\alpha \varphi$ ($\varphi$ is true with degree $\geq \alpha$, in the model $\mathcal{A}$), then, $\mathbb{I}[\mathcal{A}] \models_{[\alpha,\alpha]} \varphi$.

This paper is organized in six sections where the first one is an outline of the paper, presenting the main motivation and our goals. Then, section 2 contains some definitions and propositions which were useful to accomplish our results. Similarly, the definition of the interval constructor can be seen on section 3, as well as the application of it on classes of lattices and fuzzy connectives. We briefly present

ML logic and ML-algebras on the forth section in order to apply the interval constructor on some classes of ML-algebras on section 5, where it is found our main contributions. Finally, on the last section, some conclusions are addressed as well as suggestions for future works.

## 2 Fuzzy Connectives on Lattices

We can define lattice in two standard ways: as an algebraic structure or as a partial ordered set. However, we can say that there is a biunivocal correspondence between them (for more details refer to [8], [27]). Both constructions can be considered equivalent and whenever we mention the word lattice throughout the text, it will mean, in general, lattice by the first definition (we can also use though the second definition according to its convenience).

Concerning the algebraic definition, a nonempty set $L$ can be defined as a lattice together with two operations $\vee$ and $\wedge$ named **join** and **meet**, respectively, if it satisfies the following identities:

- $L_1$ Commutativity $x \wedge y = y \wedge x$, and $x \vee y = y \vee x$;
- $L_2$ Associativity $x \wedge (y \wedge z) = (x \wedge y) \wedge z$, and $x \vee (y \vee z) = (x \vee y) \vee z$;
- $L_3$ Idempotency $x \wedge x = x$, and $x \vee x = x$;
- $L_4$ Absorption $x \wedge (x \vee y) = x$, and $x \vee (x \wedge y) = x$.

An interesting example can be given if we let $L$ be a set of propositions, $\wedge$ denotes conjunction and $\vee$ denotes disjunction. It's well known that $L_1$ to $L_4$ are properties from propositional logic which still hold in a lattice framework.

A poset $L$ is an ordered lattice if, and only if, for every $a, b$ in $L$, both the *supremum* and *infimum* exist. Each ordered lattice determines a lattice and vice versa.

**Proposition 2.1** *Let $\mathcal{L} = \langle L, \leq \rangle$ be an ordered lattice. Then, $\mathcal{L} = \langle L, \wedge, \vee \rangle$ is a lattice, where $x \wedge y = inf\{x, y\}$ and $x \vee y = sup\{x, y\}$.*

**Proof.** See [27]. □

**Proposition 2.2** *If $\mathcal{L} = \langle L, \wedge, \vee \rangle$ is a lattice, then $\langle L, \leq_{\mathcal{L}} \rangle$ is an ordered lattice, where $x \leq_{\mathcal{L}} y$ iff $x = x \wedge y$ (or $y = x \vee y$).*

**Proof.** See [27]. □

An algebraic structure $\mathcal{L} = \langle L, \wedge, \vee, 0, 1 \rangle$ is called **bounded lattice**, if $\langle L, \wedge, \vee \rangle$ is a lattice, and the constants 0 and $1 \in L$ satisfy the following properties:

- For all $x \in L$, $x \wedge 1 = x$ and $x \vee 1 = 1$;
- For all $x \in L$, $x \wedge 0 = 0$ and $x \vee 0 = x$.

The constant 1 is called upper bound, or **top** of $\mathcal{L}$, and 0 is the lower bound, or **bottom** of $\mathcal{L}$.

Since t-norms are a relevant issue in fuzzy logics, because they model the conjunction in fuzzy logics, a good generalization of them is undoubtedly important. In [4], we obtained a lattice interpretation of the classic propositional connectives by generalizing some t-norms to bounded lattices whose definitions will be seen on the next subsections.

### *2.1 Lattice t-norm*

In [1], we defined a lattice t-norm as follows:

**Definition 2.3** Let $\mathcal{L}$ be a bounded lattice. A binary operation $*$ on $L$ is a **triangular norm** on $\mathcal{L}$, t-norm in short, if for each $x, y, z \in L$, the following properties are satisfied: commutativity: $x * y = y * x$; associativity: $x * (y * z) = (x * y) * z$; neutral element: $x * 1 = x$; and monotonicity: If $y \leq_{\mathcal{L}} z$, then $x * y \leq_{\mathcal{L}} x * z$.

Let $*_1$ and $*_2$ be t-norms on a bounded lattice $\mathcal{L}$. Then, $*_1$ is weaker than $*_2$ or, equivalently, $*_2$ is stronger than $*_1$, denoted by: $*_1 \leq_{\mathcal{L}} *_2$, if for each $x, y \in L$, $x *_1 y \leq_{\mathcal{L}} x *_2 y$. Clearly, $\leq_{\mathcal{L}}$ is a partial order on the set of t-norms on $\mathcal{L}$.

### *2.2 Lattice Implication*

A lattice implication can also be defined in a bounded lattice. As there are a lot of definitions for fuzzy implication, together with the related properties, we emphasize the ones listed below.

**Definition 2.4** Let $\mathcal{L}$ be a bounded lattice. A binary operation $\Rightarrow$ on $L$ is an **implication** on $\mathcal{L}$ if, for each $x, y, z \in L$:

- $x \leq_{\mathcal{L}} z$ implies $x \Rightarrow y \geq_{\mathcal{L}} z \Rightarrow y$, for all $x, y, z \in L$;
- $y \leq_{\mathcal{L}} z$ implies $x \Rightarrow y \leq_{\mathcal{L}} x \Rightarrow z$, for all $x, y, z \in L$;
- $0 \Rightarrow y = 1$, for all $y \in L$;
- $x \Rightarrow 1 = 1$, for all $x \in L$;
- $1 \Rightarrow 0 = 0$.

Moreover, we can obtain the residuum of the lattice t-norm $*$, as the definition below shows:

**Definition 2.5** Let $*$ be a lattice t-norm. Then, $x \Rightarrow_* y = sup\{z \in L : x * z \leq_{\mathcal{L}} y\}$, for all $x, y \in L$, is called **lattice R-implication**, or residuum of $*$ on $\mathcal{L}$.

Clearly, $\Rightarrow_*$ is an implication on $\mathcal{L}$.

### *2.3 Residuated Lattice*

The algebraic structure $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is named **residuated lattice** if:

i. $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice;

ii. $\langle L, *, 1\rangle$ is a commutative monoid;

iii. $\Rightarrow$ is a function $L^2 \to L$ such that, for all $x, y, z \in L$ : $(x * y) \leq_{\mathcal{L}} z$ iff $y \leq_{\mathcal{L}} (x \Rightarrow z)$.

The second property tells us that given an associative and commutative function $* : L^2 \to L$, then $x * 1 = x$, for all $x \in L$. Besides, in [16, prop.2] (proof in [40]) it was stated that:

**Proposition 2.6** *Let $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. Then the following properties are valid, for every $x, y$ and $z \in L$:*
*(1) $x \Rightarrow y = sup\{z \in L : x * z \leq_{\mathcal{L}} y\}$.*
*(2) $x \leq y$ iff $x \Rightarrow y = 1$.*

This proposition shows that the $\Rightarrow$ operation in a residuated lattice is completely determined from the $*$ operation.

**Proposition 2.7** *Let $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. Then the following property is valid, for every $x, y$ and $z \in L$:*
*(1) $x * y = inf\{z \in L : x \leq_{\mathcal{L}} (y \Rightarrow z)\}$.*

**Proof.** This follows from the fact that the relation between $*$ and $\Rightarrow$ is based on Galois connection [18]. □

**Proposition 2.8** *Let $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. Then $*$ is a t-norm on $L$.*

**Proof.** Once $\langle L, *, 1\rangle$ is a monoid, then it only remains to prove that it is monotonic. Let $y \leq_{\mathcal{L}} y'$, then, by the first property of definition 2.4, we have $y \Rightarrow z \geq_{\mathcal{L}} y' \Rightarrow z$. So, $\{z \in L : x \leq_{\mathcal{L}} y \Rightarrow z\} \supseteq \{z \in L : x \leq_{\mathcal{L}} y' \Rightarrow z\}$. Therefore, $inf\{z \in L : x \leq_{\mathcal{L}} y \Rightarrow z\} \leq_{\mathcal{L}} inf\{z \in L : x \leq_{\mathcal{L}} y' \Rightarrow z\}$, and by proposition 2.7, $x * y \leq_{\mathcal{L}} x * y'$. □

Note that the residuum of a t-norm $*$ is indeed an implication, thus it satisfies the properties on definition 2.4

**Proposition 2.9** *Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a bounded lattice and $*$ a t-norm on $L$. Then, $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow_*, 0, 1\rangle$ is a residuated lattice.*

**Proof.** We know $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow_*, 0, 1\rangle$ is bounded and, from definition 2.3, it is clear the associativity and the neutral element guarantee that $\langle L, *, 1\rangle$ is a commutative monoid. According to the definition of residuated lattice, it only remains to prove that $(*, \Rightarrow_*)$ form an adjoint pair. So, $x * y \leq z$ iff $y \in \{u \in L : x * u \leq_{\mathcal{L}} z\}$ iff $y \leq_{\mathcal{L}} sup\{u \in L : x * u \leq_{\mathcal{L}} z\}$ iff $y \leq_{\mathcal{L}} x \Rightarrow_* z$. □

**Corollary 2.10** *Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a bounded lattice. Then $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a residuated lattice iff $*$ is a t-norm on $L$ and $\Rightarrow = \Rightarrow_*$.*

**Proof.** Straightforward from propositions 2.6, 2.8 and 2.9. □

## 3 Interval Constructor

It is not difficult to see that an interval has a dual nature. It can be seen as a set of real numbers, but also, it can be understood as an ordered pair of real numbers.

In [9], the interval constructor was formalized (considering any partially ordered set) on the category **POSet**, as follows:

**Definition 3.1** Let $\mathcal{L} = \langle L, \leq\rangle$ be a poset. The poset $\mathbb{I}(\mathcal{L}) = \langle \mathbb{I}[L], \leq_{\mathbb{I}}\rangle$, where

- $\mathbb{I}[L] = \{[\underline{x}, \overline{x}] : \underline{x}, \overline{x} \in L \text{ and } \underline{x} \leq \overline{x}\}$
- $[\underline{x}, \overline{x}] \leq_{\mathbb{I}} [\underline{y}, \overline{y}] \Leftrightarrow \underline{x} \leq \underline{y} \text{ and } \overline{x} \leq \overline{y}$

is called the **poset of intervals of** $\mathcal{L}$.

There are also two natural functions from $\mathbb{I}[L]$ to $L$, the left and right projections $\pi_a : \mathbb{I}[L] \to L$ and $\pi_b : \mathbb{I}[L] \to L$, defined by: $\pi_a([\underline{x}, \overline{x}]) = \underline{x}$ and $\pi_b([\underline{x}, \overline{x}]) = \overline{x}$, respectively. For any interval variable $X$, $\pi_a(X)$ and $\pi_b(X)$ will be denoted, as a convention, by $\underline{x}$ and $\overline{x}$, respectively. Thus, $X = [\underline{x}, \overline{x}]$.

### *3.1 The Interval Constructor on Classes of Lattices*

In order to obtain the desired idea of an **interval residuated lattice**, $\mathbb{I}[L]$ for short, we need definition 3.1, where $\mathbb{I}[L]$ is our interval constructor $\mathbb{I}$ applied to a residuated lattice $L$. We introduced, in [37], our interval constructor on some classes of lattices, which we can briefly see below:

**Proposition 3.2** *Let $\mathcal{L} = \langle L, \wedge, \vee, 0, 1\rangle$ be a bounded lattice. Then, $\mathbb{I}[\mathcal{L}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, [0,0], [1,1]\rangle$ is also a bounded lattice, where: $X \sqcap_{\mathbb{I}} Y = [\underline{x} \wedge \underline{y}, \overline{x} \wedge \overline{y}]$, $X \sqcup_{\mathbb{I}} Y = [\underline{x} \vee \underline{y}, \overline{x} \vee \overline{y}]$, $[1,1]$ is the top of $\mathcal{L}$ and $[0,0]$ is the bottom of $\mathcal{L}$.*

**Proof.** See [37]. □

**Lemma 3.3** *Let $\langle L, *, 1\rangle$ be a commutative monoid, then $\langle \mathbb{I}[L], \mathbb{I}[*], [1,1]\rangle$ is a commutative monoid, where the operation $\mathbb{I}[*]$ is defined as follows: $X\ \mathbb{I}[*]\ Y = [\underline{x} * \underline{y}, \overline{x} * \overline{y}]$.*

**Proof.** Straightforward from definition of $\mathbb{I}[*]$ and the fact that $\langle L, *, 1\rangle$ is a commutative monoid, we know $\mathbb{I}[*]$ is commutative and it has $[1,1]$ as neutral element.□

**Lemma 3.4** *Let $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. Then, $X\ \mathbb{I}[\Rightarrow]\ Y = sup\{Z : X\ \mathbb{I}[*]\ Z \leq_{\mathbb{I}} Y\}$ where $X\ \mathbb{I}[\Rightarrow]\ Y = [\overline{x} \Rightarrow \underline{y}, \underline{x} \Rightarrow \overline{y}]$.*

**Proof.** Considering corollary 2.10, this lemma follows in an analogous way to [3, theorem 5.8]. □

**Proposition 3.5** *Let $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. Then, $\mathbb{I}[\mathcal{L}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ is also a residuated lattice, where $X\ \mathbb{I}[\Rightarrow]\ Y = [\overline{x} \Rightarrow \underline{y}, \underline{x} \Rightarrow \overline{y}]$. Moreover, $\mathbb{I}[*]$ and $\mathbb{I}[\Rightarrow]$ form an adjoint pair.*

**Proof.** By proposition 3.2, $\mathbb{I}[\mathcal{L}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, [0,0], [1,1]\rangle$ is a bounded lattice and by lemma 3.3, $\langle \mathbb{I}[L], \mathbb{I}[*], [1,1]\rangle$ is a commutative monoid. Then, it would only remain to prove that $\mathbb{I}[\Rightarrow]$ is a function $\mathbb{I}[L]^2 \to \mathbb{I}[L]$ such that, for all $X, Y, Z \in \mathbb{I}[L] : (X\ \mathbb{I}[*]\ Y) \leq_{\mathbb{I}} Z$ iff $Y \leq_{\mathbb{I}} (X\ \mathbb{I}[\Rightarrow]\ Z)$. However, by lemma 3.4 this proof is analogous to the one used to prove proposition 2.9, as $\mathcal{L}$ is a residuated lattice. □

### *3.2 The Interval Constructor on Fuzzy Connectives*

In [1], we obtained an important result showing how to transform an arbitrary t-norm, on a bounded lattice, into a t-norm on its interval bounded lattice.

**Proposition 3.6** *Let $*$ be a t-norm on the bounded lattice $L$. Then $\mathbb{I}[*] : \mathbb{I}[L]^2 \to \mathbb{I}[L]$ defined by $X\ \mathbb{I}[*]\ Y = [\underline{x} * \underline{y}, \overline{x} * \overline{y}]$ is a t-norm on the bounded lattice $\mathbb{I}[L]$.*

**Proof.** It is easy to see that the commutativity, monotonicity and neutral element ($[1,1]$) properties of $\mathbb{I}[*]$ follow straightforward from the same properties of the t-norm $*$. $\square$

An arbitrary implication $\Rightarrow$, on a bounded lattice, can be transformed as well, into an implication on its interval bounded lattice, represented here by $\mathbb{I}[\Rightarrow]$. Despite of the many different definitions for fuzzy implications and their related properties, we will only consider some of them. The following proposition shows us how to achieve $\mathbb{I}[\Rightarrow]$, based on the work of Bedregal et. al. ([2]).

Another implication on the bounded lattice $\mathbb{I}[L]$, proposed in [3], can be seen below:

**Proposition 3.7** *Let $\Rightarrow$ be a fuzzy implication on the bounded lattice $L$. Then $\mathbb{I}[\Rightarrow]$, defined by $X\ \mathbb{I}[\Rightarrow]\ Y = [\overline{x} \Rightarrow \underline{y}, \underline{x} \Rightarrow \overline{y}]$ is a fuzzy implication on the bounded lattice $\mathbb{I}[L]$.*

**Proof.** See [2]. $\square$

**Proposition 3.8** *Let $*$ be a t-norm on a bounded lattice $\mathcal{L}$, then $(X \Rightarrow_{\mathbb{I}[*]} Y = sup\{Z \in \mathbb{I}[\mathcal{L}] : X\ \mathbb{I}[*]\ Z \leq_{\mathbb{I}} Y\}$ is a R-implication on $\mathbb{I}[\mathcal{L}]$. Moreover, $\Rightarrow_{\mathbb{I}[*]} = \mathbb{I}[\Rightarrow]$*

**Proof.** Analogous to [3, Theorem 6.9]. $\square$

Let $\mathcal{L} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ be a residuated lattice. It is clear, by the corollary 2.10, that $(*, \Rightarrow)$ form an adjoint pair [14]. Then, to obtain the adjoint pair, after the addition of the interval constructor, is straightforward: $(\mathbb{I}[*], \mathbb{I}[\Rightarrow])$.

With these results, one important conclusion, obtained in [3], was the fact that the diagram shown in figure 1 commutes. This diagram represents the idea of existing two possible ways to obtain $\mathbb{I}[\Rightarrow]$.

$$\begin{tikzcd} * \arrow[r] \arrow[d] & \Rightarrow \arrow[d] \\ \mathbb{I}[*] \arrow[r] & \mathbb{I}[\Rightarrow] = \Rightarrow_{\mathbb{I}[*]} \end{tikzcd}$$

Fig. 1. Commutative diagram relating $*, \Rightarrow, \mathbb{I}[*]$ and $\mathbb{I}[\Rightarrow]$.

On one hand, we can add the interval constructor directly to $\Rightarrow$. On the other hand, we can first add it to the t-norm $*$ and then, from $\mathbb{I}[*]$, we can obtain $\mathbb{I}[\Rightarrow]$. It was proved in proposition 3.8 that $\mathbb{I}[\Rightarrow]$ and $\Rightarrow_{\mathbb{I}[*]}$ are equal.

Most of the definitions and propositions given so far are fundamental to prove that some classes of ML-algebras are preserved by the interval constructor. Monoidal Logic and its algebraic counterpart, namely ML-algebra, will be introduced on the next section.

## 4 Monoidal Logic and ML-algebras

Monoidal Logic, ML for short, introduced by Höhle in [24], whose algebraic counterpart is the class of residuated lattices, gave a common framework to several first order non-classical logics, such as Linear logic, Łukasiewicz logic, among others. This logic was built up from the following primitive connectives: $*, \rightarrow, \wedge$ and $\vee$, and the truth constant $\overline{0}$ [3]. We can note that the connectives $\wedge$ and $\vee$ can not be defined from the others, and this is the reason why they were introduced as primitive connectives.

Some extensions of this logic seem quite interesting because there is a sort of interaction between these extensions and other logics. ML extensions include: MTL (Monoidal T-norm based Logic), introduced by Esteva and Godo in [14] (which is intended to cope with the tautologies of left-continuous t-norms and their residua); IMTL (Involutive Monoidal t-norm logic), WNM (Weak Nilpotent Minimum logic) and NM (Nilpotent Minimum logic), which are extensions of MTL.

In figure 2, there is an interesting diagram of logics (and their axioms) and the relationships between ML and MTL with its extensions, BL (Basic Fuzzy logic, that is the many-valued residuated logic, introduced by Hájek in [23]), Łukasiewicz logic, and Affine Multiplicative Linear logic, AMALL for short, a propositional fragment of Girard's Linear logic, in [19] and [38]. The arrows indicate the extensions labeled with the axioms added.

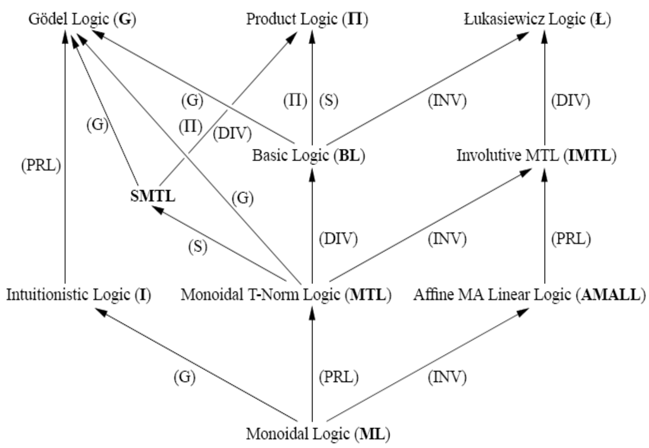


Fig. 2. Relationships between fuzzy logics.

Some authors prefer to use the bounded integral commutative lattice terminology instead of residuated lattice (e.g. [15]), but once both structures are essentially the same [16], we consider the following definition (based on [15] and [25]) for a ML-algebra:

**Definition 4.1** $\mathcal{A} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is a **ML-algebra** if:

(i) $\langle L, \wedge, \vee\rangle$ is a bounded lattice with order $\leq_{\mathcal{L}}$, top element 1 and bottom 0;

(ii) $\langle L, *, 1\rangle$ is a commutative semigroup with unit element 1;

(iii) $*$ and $\Rightarrow$ form an adjoint pair (i.e. $y \leq_{\mathcal{L}} x \Rightarrow z$ iff $x*y \leq_{\mathcal{L}} z$ , for all $x, y, z \in L$).

[3] Höhle uses negation as primitive connective, in the original formal system, in [24]. However, we prefer to use the axiomatic system, given in [21], which is equivalent.

Thus, ML-algebras are residuated lattices.

The unary operation $\neg$, defined by $\neg x =_{def} x \Rightarrow 0$, is called **negation**. And, we can also define the operation: $x \oplus y =_{def} \neg(\neg x * \neg y)$, called **disjunction**.

Many classes of ML-algebras can be obtained by including some properties which, in the end, means that refinements of ML-algebras, suitable for fuzzy logics, are defined by the addition of specific axioms to the Monoidal Logic. In this way, we have the following classes of residuated lattices, resulted from the following definitions:

**Definition 4.2** A ML-algebra $\langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ is

(INV) **dualizing** iff $\neg\neg x = x$, for all $x \in L$;

(G) **idempotent** iff $x * x = x$, for all $x \in L$;

(PRL) **prelinear** iff $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$, for all $x, y \in L$;

(DIV) **divisible** iff $x \wedge y = x * (x \Rightarrow y)$, for all $x, y \in L$;

(S) **weakly contracting** iff $x \wedge \neg x = 0$, for all $x \in L$;

($\prod$) **weakly cancellative** iff $\neg\neg x \leq (x \Rightarrow (x * y)) \Rightarrow y$, for all $x, y \in L$.

These classes can receive special names, such as the dualizing ML-algebra, which is called AMALL-algebra, for short (standing for the affine multiplicative additive fragment of Linear logic). The class MTL-algebra (introduced in [14]), which is also called prelinear ML-algebra, is a subclass of ML-algebra where $*$ is a left-continuous t-norm and $\Rightarrow$ is the *residumm* of $*$, forming the adjoint pair $(*, \Rightarrow)$.

Observing table (3) we see different classes of ML-algebras obtained after the addition of one of the properties mentioned above. The most famous members of these classes are: MTL-algebra, IMTL-algebra, SMTL-algebra, each one having the left-continuous t-norm $*$ and its *residuum* $\Rightarrow$, and BL-algebra, Ł-algebra, G-algebra, $\prod$-algebra, whose t-norms are continuous with the correspondent *residua*. More specifically, the continuous t-norm $*$ of Ł-algebra, G-algebra and $\prod$-algebra are the Łukasiewicz, Gödel and Product t-norms, respectively.

| *Name* | *Class of Residuated Lattices* | *Property included* |
|---|---|---|
| AMALL-algebra | Dualizing ML-algebra | ML + (INV) |
| MTL-algebra | Prelinear ML-algebra | ML + (PRL) |
| IMTL-algebra | Dualizing MTL-algebra | MTL + (INV) |
| SMTL-algebra | Weakly contracting MTL-algebra | MTL + (S) |
| BL-algebra | Divisible MTL-algebra | MTL + (DIV) |
| Ł-algebra | Dualizing BL-algebra | BL + (INV) |
| G-algebra | Idempotent BL-algebra | BL + (G) |
| $\prod$-algebra | Weakly contracting and weakly cancellative BL-algebra | BL + (S) + ($\prod$) |

Table 3: Classes of residuated lattices and their respective names.

In order to show the relationships between some subclasses of ML-algebras and the interval constructor $\mathbb{I}$ we will draw a new diagram, presenting the subclasses which are preserved by the interval constructor.

## 5 The Interval Constructor on ML-algebras

A ML-algebra $(L, \wedge, \vee, *, \Rightarrow, 0, 1)$ is a residuated lattice such that $(*, \Rightarrow)$ form an adjoint pair. We will prove that the class of ML-algebra is preserved by the interval constructor. Formally:

**Proposition 5.1** *Let* $\mathcal{A} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be a ML-algebra. Then* $\mathbb{I}[\mathcal{A}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also a ML-algebra.*

**Proof.** Straightforward from proposition 3.5 once ML-algebras and residuated lattices coincide. $\square$

The interval constructor $\mathbb{I}$, on some subclasses of ML-algebras, can also be done by the addition of some properties to certain classes of residuated lattices. In fact, the different classes of residuated lattices, seen on the section before, are ML-algebras (or other subclasses of them) which satisfy a certain property (i.e., involution, prelinearity, divisibility, etc.). So, for instance, if a ML-algebra satisfies the dualizing property (INV), we say it is an AMALL-algebra. And the correspondent algebra with the interval constructor is obtained as follows:

**Proposition 5.2** *Let* $\mathcal{AM} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be an AMALL-algebra. Then,* $\mathbb{I}[\mathcal{AM}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also an AMALL-algebra.*

**Proof.** By proposition 5.1, $\mathbb{I}[\mathcal{AM}]$ is already a ML-algebra. Then, by the definition of AMALL-algebra, it only remains to prove that: $(X\ \mathbb{I}[\Rightarrow]\ [0,0])\ \mathbb{I}[\Rightarrow]\ [0,0] = X$, for all $X \in \mathbb{I}[\mathcal{L}]$.

By the definition of $\mathbb{I}[\Rightarrow]$:

$$
\begin{aligned}
(X\ \mathbb{I}[\Rightarrow]\ [0,0])\ \mathbb{I}[\Rightarrow]\ [0,0] &= [(\underline{x} \Rightarrow 0) \Rightarrow 0, (\overline{x} \Rightarrow 0) \Rightarrow 0];\ \text{lemma 3.4.}\\
&= [\neg\neg\underline{x}, \neg\neg\overline{x}];\ \text{by definition of } \neg.\\
&= [\underline{x}, \overline{x}];\ \text{by property (INV).}\\
&= X.
\end{aligned}
$$

$\square$

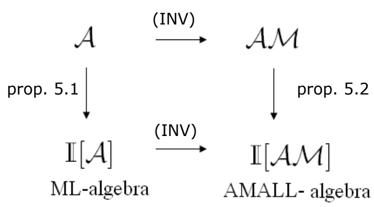


Fig. 3. $\mathbb{I}[\mathcal{AM}]$; a dualizing $\mathbb{I}[\mathcal{A}]$.

Hence, the dualizing condition holds, which means $\mathbb{I}[\mathcal{AM}]$ is an AMALL-algebra.

Besides, by propositions 5.2 and 3.5, and observing table 3, we may conclude the diagram shown in figure 3 commutes.

Using a similar idea, if an AMALL-algebra satisfies the following property (PRL): $(x \Rightarrow y) \vee (y \Rightarrow x)$, we say it is an IMTL-algebra. Then:

**Proposition 5.3** *Let* $\mathcal{IM} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be an IMTL-algebra. Then,* $\mathbb{I}[\mathcal{IM}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also an IMTL-algebra.*

**Proof.** By proposition 5.2, $\mathbb{I}[\mathcal{IM}]$ is already an AMALL-algebra. So, it only remains to prove that $X\ \mathbb{I}[\Rightarrow]\ Y \sqcup_{\mathbb{I}} Y\ \mathbb{I}[\Rightarrow]\ X = [1,1]$.

$$
\begin{aligned}
X\ \mathbb{I}[\Rightarrow]\ Y \sqcup_{\mathbb{I}} Y\ \mathbb{I}[\Rightarrow]\ X &= [\overline{x} \Rightarrow \underline{y}, \underline{x} \Rightarrow \overline{y}] \sqcup_{\mathbb{I}} [\overline{y} \Rightarrow \underline{x}, \underline{y} \Rightarrow \overline{x}]; \text{ lemma 3.4.}\\
&= [(\overline{x} \Rightarrow \underline{y}) \vee (\overline{y} \Rightarrow \underline{x}), (\underline{x} \Rightarrow \overline{y}) \vee (\underline{y} \Rightarrow \overline{x})]; \text{ prop. 3.2.}\\
&= [((\overline{x} \Rightarrow \underline{y}) \vee (\overline{y} \Rightarrow \underline{x})) \wedge ((\underline{x} \Rightarrow \overline{y}) \vee (\underline{y} \Rightarrow \overline{x})),\\
&\quad ((\overline{x} \Rightarrow \underline{y}) \vee (\overline{y} \Rightarrow \underline{x})) \vee ((\underline{x} \Rightarrow \overline{y}) \vee (\underline{y} \Rightarrow \overline{x}))]; \text{ def. 2.4.}\\
&= [((\overline{x} \Rightarrow \underline{y}) \wedge (\underline{x} \Rightarrow \overline{y})) \vee ((\overline{y} \Rightarrow \underline{x}) \wedge (\underline{x} \Rightarrow \overline{y})) \vee\\
&\quad ((\overline{x} \Rightarrow \underline{y}) \wedge (\underline{y} \Rightarrow \overline{x})) \vee ((\overline{y} \Rightarrow \underline{x}) \wedge (\underline{y} \Rightarrow \overline{x})),\\
&\quad ((\overline{x} \Rightarrow \underline{y}) \vee (\underline{y} \Rightarrow \overline{x})) \vee ((\underline{x} \Rightarrow \overline{y}) \vee (\overline{y} \Rightarrow \underline{x}))]; \text{ distrib.}\\
&= [((\overline{x} \Rightarrow \underline{y}) \wedge (\underline{x} \Rightarrow \overline{y})) \vee 1 \vee 1 \vee ((\overline{y} \Rightarrow \underline{x}) \wedge (\underline{y} \Rightarrow \overline{x})),\\
&\quad 1 \wedge 1]; \text{ proposition 2.6.}\\
&= [1,1].
\end{aligned}
$$

□

It is clear $\mathbb{I}[\mathcal{IM}]$ is an IMTL-algebra. Besides, by propositions 5.3 and 3.5, and observing table 3, we may also conclude the diagram shown on figure 4 commutes.

$$
\begin{tikzcd}
\mathcal{AM} \arrow[r, "\text{(PRL)}"] \arrow[d, "\text{prop. 5.2}"'] & \mathcal{IM} \arrow[d, "\text{prop. 5.3}"] \\
\underset{\text{AMALL- algebra}}{\mathbb{I}[\mathcal{AM}]} \arrow[r, "\text{(PRL)}"] & \underset{\text{IMTL-algebra}}{\mathbb{I}[\mathcal{IM}]}
\end{tikzcd}
$$

Fig. 4. $\mathbb{I}[\mathcal{IM}]$; a prelinear $\mathbb{I}[\mathcal{AM}]$.

If a ML-algebra satisfies the prelinearity (PRL) property, we say it is a MTL-algebra.

**Proposition 5.4** *Let* $\mathcal{M} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be a MTL-algebra. Then,* $\mathbb{I}[\mathcal{M}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also a MTL-algebra.*

**Proof.** By proposition 5.1, $\mathbb{I}[\mathcal{M}]$ is a ML algebra, so according to the definition of MTL-algebra it would only remain to prove that: $X\ \mathbb{I}[\Rightarrow]\ Y \sqcup_{\mathbb{I}} Y\ \mathbb{I}[\Rightarrow]\ X = [1,1]$.

But this proof is analogous to the one on the previous proposition (5.3). So, $\mathbb{I}[\mathcal{M}]$ is a MTL-algebra indeed. □

Observe that we can also use MTL-algebra $\mathbb{I}[\mathcal{M}]$ to obtain an IMTL-algebra $\mathbb{I}[\mathcal{IM}]$. Analyzing figure 2 more carefully, we can see the Involutive MTL is obtained via MTL after the addition of the (INV) axiom.

**Proposition 5.5** *Let* $\mathcal{IM} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be an IMTL-algebra. Then,* $\mathbb{I}[\mathcal{IM}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also an IMTL-algebra.*

**Proof.** By proposition 5.4, $\mathbb{I}[\mathcal{M}]$ is a MTL algebra, so, according to the definition of IMTL-algebra, it would only remain to prove that: $(X\ \mathbb{I}[\Rightarrow]\ [0,0])\ \mathbb{I}[\Rightarrow]\ [0,0] = X$ holds. But this proof is analogous to the one on proposition (5.2). Therefore, $\mathbb{I}[\mathcal{M}]$ is an IMTL-algebra indeed. □

Actually, by propositions 5.5 and 3.5, and observing table 3, we may conclude the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "\text{(INV)}"] \arrow[d, "\text{prop. 5.4}"'] & \mathcal{IM} \arrow[d, "\text{prop. 5.5}"] \\
\mathbb{I}[\mathcal{M}] \arrow[r, "\text{(INV)}"] & \mathbb{I}[\mathcal{IM}] \\
\text{MTL-algebra} & \text{IMTL-algebra}
\end{tikzcd}$$

Fig. 5. $\mathbb{I}[\mathcal{IM}]$; a dualizing $\mathbb{I}[\mathcal{M}]$.

Finally, if a MTL-algebra satisfies the weakly contracting property (S), we say it is a SMTL-algebra. Then:

**Proposition 5.6** *Let* $\mathcal{SM} = \langle L, \wedge, \vee, *, \Rightarrow, 0, 1\rangle$ *be a SMTL-algebra. Then,* $\mathbb{I}[\mathcal{SM}] = \langle \mathbb{I}[L], \sqcap_{\mathbb{I}}, \sqcup_{\mathbb{I}}, \mathbb{I}[*], \mathbb{I}[\Rightarrow], [0,0], [1,1]\rangle$ *is also a SMTL-algebra.*

**Proof.** Considering proposition 5.4, we conclude $\mathbb{I}[\mathcal{SM}]$ is a MTL-algebra. And, by the definition of SMTL-algebra, we need to prove that: $X \sqcap_{\mathbb{I}} (X\ \mathbb{I}[\Rightarrow]\ [0,0]) = [0,0]$, for all $X \in \mathbb{I}[\mathcal{L}]$. Hence:

$$\begin{aligned}
X \sqcap_{\mathbb{I}} (X\ \mathbb{I}[\Rightarrow]\ [0,0]) &= [\underline{x}, \overline{x}]\ \sqcap_{\mathbb{I}} [\overline{x} \Rightarrow 0, \underline{x} \Rightarrow 0]; \text{ lemma 3.4.} \\
&= [\underline{x} \wedge (\overline{x} \Rightarrow 0), \overline{x} \wedge (\underline{x} \Rightarrow 0)]; \text{ proposition 3.2.} \\
&= [(\underline{x} \wedge (\overline{x} \Rightarrow 0)) \wedge (\overline{x} \wedge (\underline{x} \Rightarrow 0)), \\
&\quad (\underline{x} \wedge (\overline{x} \Rightarrow 0)) \vee (\overline{x} \wedge (\underline{x} \Rightarrow 0))]; \text{ definition 2.4.} \\
&= [(\underline{x} \wedge (\underline{x} \Rightarrow 0)) \wedge (\overline{x} \wedge (\overline{x} \Rightarrow 0)), \\
&\quad (\underline{x} \vee \overline{x}) \wedge ((\overline{x} \Rightarrow 0) \vee \overline{x}) \wedge (\underline{x} \vee (\underline{x} \Rightarrow 0)) \wedge \\
&\quad ((\overline{x} \Rightarrow 0) \vee (\underline{x} \Rightarrow 0))]; \text{ distributivity.} \\
&= [0 \wedge 0, (\underline{x} \vee \overline{x}) \wedge 0 \wedge 0 \wedge ((\overline{x} \Rightarrow 0) \vee (\underline{x} \Rightarrow 0))]; \text{ by (S).} \\
&= [0, 0].
\end{aligned}$$

□

Therefore, $\mathbb{I}[\mathcal{SM}]$ is a SMTL-algebra. And by propositions 5.6 and 3.5, and loooking at table 3, we may conclude the diagram on figure 6 commutes.

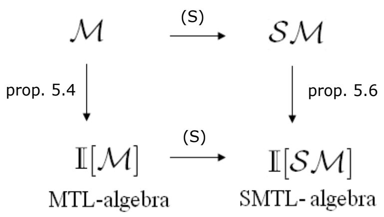


Fig. 6. $\mathbb{I}[\mathcal{SM}]$; a weakly contracting $\mathbb{I}[\mathcal{M}]$.

The relationships between ML-algebra ($\mathbb{I}[\mathcal{A}]$), AMALL-algebra ($\mathbb{I}[\mathcal{AM}]$), MTL-algebra ($\mathbb{I}[\mathcal{M}]$), IMTL-algebra ($\mathbb{I}[\mathcal{IM}]$) and SMTL-algebra ($\mathbb{I}[\mathcal{SM}]$) can be seen on figure 7.

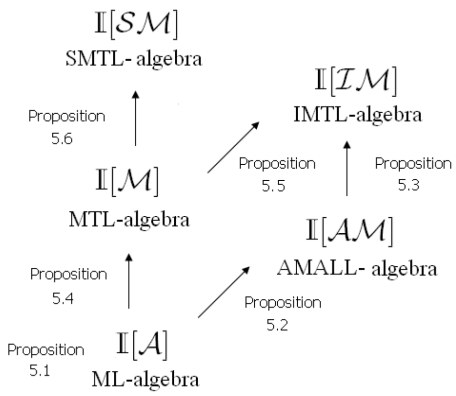


Fig. 7. Relationships between ML-algebra and its subclasses with the interval constructor $\mathbb{I}$.

Summarizing, it is clear that, from proposition 5.1, we obtain $\mathbb{I}[\mathcal{A}]$, and from proposition 5.2, we achieve $\mathbb{I}[\mathcal{AM}]$. Then, from proposition 5.3, we get $\mathbb{I}[\mathcal{IM}]$, from proposition 5.4, we reach $\mathbb{I}[\mathcal{M}]$, and from proposition 5.5, we obtain $\mathbb{I}[\mathcal{IM}]$ via $\mathbb{I}[\mathcal{M}]$. Finally, from proposition 5.6, we have $\mathbb{I}[\mathcal{SM}]$.

We can also construct another diagram considering all the possible ways to achieve classes of ML-algebras by the addition of the dualizing (INV), prelinear (PRL) and weakly contracting (S) properties as showed in figure 8.

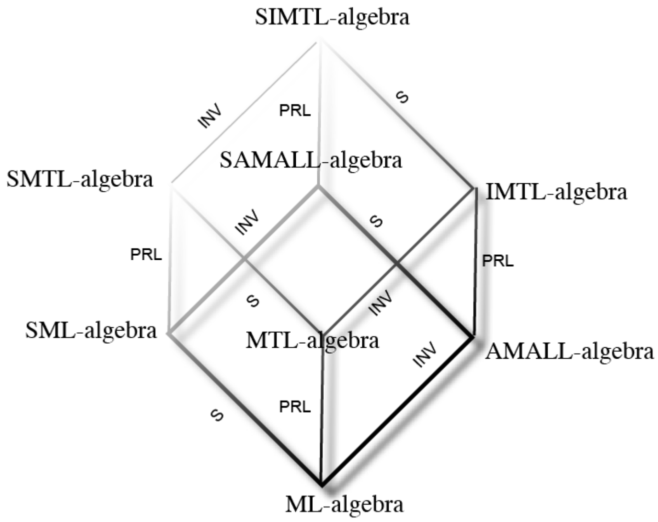


Fig. 8. Classes of ML-algebras by including properties INV, PRL and S and the interval constructor $\mathbb{I}$.

It is possible to draw some conclusions by observing figure 8 more carefully. For instance, it is clear how we achieve the well-known results: IMTL = AMALL + (PRL) = MTL + (INV) and SMTL = MTL + (S). Besides, possible new ones would be:

- SIMTL = SMTL + (INV) = IMTL + (S),
- SMTL = SML + (PRL),
- SAMALL = AMALL + (S) = SML + (INV) and
- SML = ML + (S).

Clearly, SML-algebra, SAMALL-algebra and SIMTL-algebra would be new classes of ML-algebras after the addition of the properties mentioned previously. Actually, once we have already proved that the interval constructor $\mathbb{I}$ preserves the properties (INV), (PRL) and (S) - propositions 5.2, 5.3 and 5.6 - then it is easy to see that these classes are closed on the interval constructor.

## 6 Concluding Remarks

This work was a first step to apply the interval constructor $\mathbb{I}$ on classes of ML-algebras. In order to do that, we used some interesting results in ML theory itself, for instance, proposition 2.7 and corollary 2.10. Some other related studies, which had been done over the last three years, namely [4], [5], [1], [10], [6] and [37] were also relevant to achieve some of our goals.

In this paper, we prove that some classes of ML-algebras are preserved by the interval constructor. The importance of these results is that each of them determine important classes of fuzzy logics, giving us the possibility to deal with membership degrees represented by intervals in the unit interval $[0, 1]$, which can be used in interval computations allowing an automatic and rigid control of errors (aiming to decrease them), as well as, whenever we deal with many expert's opinions to determine the membership degree without discarding none of their opinions.

For further works we can suggest the addition of other axioms in order to enrich figure 2 and obtain other fuzzy logics and consequently other classes of ML-algebras. Another future work would be the inclusion of other logics (even modal logic) within this context. We could also include automorphisms and verify if the classes of ML-algebras are closed on the action of this automorphism (using automorphisms in a similar way as it was done in [7]) and relate it to the interval constructor over these classes. Besides , if we pay more attention to the diagram of logics and their relating axioms (figure 2), there are lacking arrows (with their respective axioms needed) which would establish other connections between the fuzzy logics. Moreover, it would also be interesting to analyze the behavior of $\mathbb{I}$ on those logics.

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