On Interval Fuzzy Negations

Benjamín Callejas Bedregal

Laboratory of Logic, Language, Information, Theory and Applications – LoLITA
Department of Informatics and Applied Mathematics – DIMAp
Federal University of Rio Grande do Norte – UFRN
59.072-970 Natal, RN, Brazil.
bedregal@dimap.ufrn.br

Abstract

There exists infinitely many ways to extend the classical propositional connectives to the set $[0, 1]$ such that the behavior in its extremes is as in the classical logic. Still, it is a consensus that it is not sufficient, demanding that these extensions also preserve some minimal logical properties of the classical connectives. Thus, the notions of t-norms, t-conorms, fuzzy negations, and fuzzy implications were introduced.

In previous works, the authors generalize these notions to the set $\mathbb{U} = \{[a, b]/0 \leq a \leq b \leq 1\}$ and provided canonical constructions to obtain, for example, an interval t-norm which is the best interval representation of a t-norm.

In this paper, we considered the notion of interval fuzzy negation and generalized, in a natural way, several notions related with fuzzy negations, such as equilibrium point and negation-preserving automorphism, and we show that the main properties of these notions are preserved for the proposed interval generalizations.

Key words: Interval representations, fuzzy negations, equilibrium point, automorphisms.

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1 Introduction

Intelligent computational systems using fuzzy logics, i.e. fuzzy systems, are efficient to deal with uncertain information and therefore with approximate reasoning [9]. For that, to each variable (linguistic terms) in the system the membership degree is considered to each possible value that the variable could take (universe of discourse). The membership degrees are usually obtained from an expert evaluation. Moreover, an expert is able to determine his belief degree with certain level of precision, for example he could easily distinguish between his degree of belief 0.8 and 0.9, but it would be hard to distinguish between 0.8 and 0.8001 [54], i.e. while more precision is considered in the belief degree, more difficulty the expert will have to determine his belief degree. An alternative is to consider interval mathematics, whose main objective is the automatic and rigorous control of digital error of numerical computations and therefore it is adequate to deal with the imprecision of the input values and those caused by the roundoff errors which occur during the computation [37,38,2]. Thus, fuzzy logics joined with interval mathematics could allow to deal with uncertainty as well as with imprecision. Several ways to unite these two areas have been researched, see for example [15,52,16,30,32,36,31,20]. In [33], Weldom Lodwick points out four relationships between fuzzy set theory and interval analysis. The fourth one uses intervals as degree of membership of fuzzy sets with the goal of addressing the uncertainty associated with digital computers. In this approach the membership degree of each object is a subinterval of the unit interval $U = [0, 1]$ and therefore is also adequate to deal with the imprecision of a specialist in providing an exact value to measure membership uncertainty.

There exists infinitely many ways to extend the classical propositional connectives to the set U such that the behavior in its extremes is as in the classical logic. Still, it is a consensus that it is not sufficient, demanding that these extensions also preserve some logical properties of the classical connectives. From the seminal work of Lotfi A. Zadeh in [55] several approaches had been proposed for fuzzy negations. Although Zadeh fuzzy negation $C(x) = 1 - x$ is the most used in fuzzy systems, there are important classes of fuzzy negation proposed with different motivations. The class of Sugeno complement is obtained from a kind of special measures defined by Michio Sugeno himself in [49], and Ronald Yager class of fuzzy negations, which results from the fuzzy unions by requiring that $N(x) \vee x = 1$ for each $x \in U$. Both, can derive most of the fuzzy negations which are used in the practice [43]. Nevertheless, other different fuzzy negations were defined in the end of the 70's and beginning of the 80's, for example, in [34,51,18,25,42]. The axiomatic definition as it is known today for the fuzzy negation can be found in [25].

On the other hand, the notion of fuzzy negations for the interval value universe, i.e. $\mathbb{U} = \{[a, b]/0 \leq a \leq b \leq 1\}$, is newer. Several interval valued fuzzy negations have been proposed, see for example [23,21,41,13,5,11]. In this work the notion of interval fuzzy negation of [5] is considered which is a restriction of the similar notion used in [13,11] by considering the monotonicity condition from two different interval orders. It is proved that several properties of fuzzy negations and their strict and strong subclasses are preserved by their interval counterpart. In this sense, we consider interval versions of the notions of equilibrium point (or fixed point) of fuzzy negations investigated by [25,30,53] among others; automorphisms and the characterization theorems of Enric Trillas [51] and János Fodor [19] of strong and strict fuzzy negations, respectively, as well as their action on fuzzy negations; and the notion of negation-preserving automorphism introduced by Mirko Navara in [40] and a generalization of these concepts by considering an arbitrary strong fuzzy negation instead of Zadeh fuzzy negation.

2 Fuzzy Negations and automorphisms

In order to make this paper self-contained we will present the main definitions and properties of fuzzy negations, automorphisms and other correlated concepts. More details can be found by the readers in texts such as [51,19,30,40,28,10,35].

2.1 Fuzzy Negation

A function $N : U \rightarrow U$, where U denotes the unit interval $[0, 1]$, is a **fuzzy negation** if

- N1: $N(0) = 1$ and $N(1) = 0$.
- N2: If $x \leq y$ then $N(y) \leq N(x)$, $\forall x, y \in U$.

Fuzzy negations are **strict** if it satisfies the following properties

- N3: N is continuous,
- N4: If $x < y$ then $N(y) < N(x)$, $\forall x, y \in U$.

Fuzzy negations satisfying the **involutive** property, i.e.

- N5: $N(N(x)) = x$, $\forall x \in U$,

are called **strong fuzzy negations**. Notice that each strong fuzzy negation is strict but the reverse is not true. For example, the fuzzy negation $N(x) = 1-x^2$ is strict but not strong.

Notice that if N is a strong fuzzy negation, then $N = N^{-1}$.

An **equilibrium point** of a fuzzy negation N is a value $e \in U$ such that $N(e) = e$.

Remark 2.1 *Let N be a fuzzy negation. If e is an equilibrium point for N then by antitonicity of N for each $x \in U$, if $x \leq e$ then $e \leq N(x)$ and if $e \leq x$ then $N(x) \leq e$.*

Remark 2.2 *Let N be a fuzzy negation. If e is an equilibrium point for N and if $x \leq N(x)$ then $x \leq e$ and if $N(x) \leq x$ then $e \leq x$.*

George Klir and Bo Yuan in [30] proved that all fuzzy negations have at most one equilibrium point and so if a fuzzy negation N has an equilibrium point then it is unique. For example, the strict fuzzy negation $N(x) = 1 - x^2$ has $\frac{\sqrt{5}-1}{2} \cong 0.618034$ as the unique equilibrium point. However, not all fuzzy negations have an equilibrium point, for example the fuzzy negation $N_\perp$, defined below has no equilibrium point.

$$N_\perp(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Remark 2.3 *Let N be a strict (strong) fuzzy negation. Then by continuity, N has an equilibrium point. As noted above, its equilibrium point is unique.*

Remark 2.4 *Let $e \in U$. Then there exists infinitely many fuzzy negations having e as equilibrium point. For example, if N is a strong fuzzy negation then the function $N' : U \rightarrow U$, defined as*

$$N'(x) = \begin{cases} N(\frac{N(e)x}{e}) & \textit{if } x \leq e \\ \frac{N(x)e}{N(e)} & \textit{if } x > e, \end{cases}$$

is a strict fuzzy negation such that $N'(e) = e$.

Analogously to a t-norm, it is also possible to establish a partial order on the fuzzy negations in a natural way, i.e. given two fuzzy negations N_1 and N_2 we say that $N_1 \leq N_2$ if for each $x \in U$, $N_1(x) \leq N_2(x)$.

Proposition 2.1 *Let N_1 and N_2 be fuzzy negations such that $N_1 \leq N_2$. Then if e_1 and e_2 are the equilibrium points of N_1 and N_2, respectively, then $e_1 \leq e_2$.*

Proof: Let e_1 and e_2 the equilibrium points of N_1 and N_2 respectively. Suppose that $e_2 < e_1$ then $e_1 \leq N_1(e_2)$. Thus, because $N_1 \leq N_2$, $e_1 \leq N_1(e_2) \leq N_2(e_2) = e_2$ which is a contradiction. Therefore, $e_1 \leq e_2$. ∎

Clearly, for any fuzzy negation N,

$$N_{\perp} \leq N \leq N_{\top} \tag{1}$$

where

$$N_{\top}(x) = \begin{cases} 0 & \text{if } x = 1 \\ 1 & \text{if } x < 1 \end{cases}$$

Notice that neither $N_{\perp}$ nor $N_{\top}$ are strict. Then, it is natural to ask, there exists a lesser and a greater strict (strong) fuzzy negation?

In the next subsection we will answer this question.

2.2 Automorphisms

A mapping $\rho : U \longrightarrow U$ is an **automorphism** if it is bijective and monotonic, i.e. $x \leq y \;\Rightarrow\; \rho(x) \leq \rho(y)$ [29,40]. An equivalent definition was given in [10], where automorphisms are continuous and strictly increasing functions $\rho : U \longrightarrow U$ such that $\rho(0) = 0$ and $\rho(1) = 1$. Automorphisms are closed under composition, i.e. if ρ and ρ' are automorphisms then $\rho \circ \rho'(x) = \rho(\rho'(x))$ is also an automorphism. The inverse of an automorphism is also an automorphism. Thus, $(Aut(U), \circ)$, where $Aut(U)$ is the set of all automorphisms, is a group, with the identity function being the neutral element and ρ^{-1} being the inverse of ρ [21].

Let ρ be an automorphism and N be a fuzzy negation. The **action of ρ on N**, denoted by N^{ρ}, is defined as follows

$$N^{\rho}(x) = \rho^{-1}(N(\rho(x))) \tag{2}$$

Notice that, if e is the equilibrium point of a fuzzy negation N, then $\rho^{-1}(e)$ is the equilibrium point of N^{ρ}.

Proposition 2.2 *Let $N : U \longrightarrow U$ be a fuzzy negation and $\rho : U \longrightarrow U$ be an automorphism. Then N^{ρ} is also a fuzzy negation. Moreover, if N is strict (strong) then N^{ρ} is also strict (strong).*

Proof: Let $x, y \in U$.

- $N1$: Trivially, $N^{\rho}(0) = \rho^{-1}(N(\rho(0))) = \rho^{-1}(N(0)) = 1$.

- $N2$: If $x \leq y$ then $\rho(x) \leq \rho(y)$. Thus, by $N2$, $N(\rho(y)) \leq N(\rho(x))$ and so $\rho^{-1}(N(\rho(y))) \leq \rho^{-1}(N(\rho(x)))$. So, $N^\rho(y) \leq N^\rho(x)$.
- $N3$: Composition of continuous functions is also continuous.
- $N4$: Analogous to $N2$.
- $N5$: $N^\rho(N^\rho(x)) = \rho^{-1}(N(\rho^{-1}(\rho(N(\rho(x)))))) = \rho^{-1}(N(N(\rho(x))))$ $= \rho^{-1}(\rho(x)) = x$.

■

Proposition 2.3 *Let N be a strict (strong) fuzzy negation and the automorphism $\rho(x) = x^2$. Then, $N < N^\rho$ and $N^{\rho^{-1}} < N$.*

Proof: Note that $\rho^{-1}(x) = \sqrt{x}$. Since $x^2 < x$ for each $x \in (0,1)$, then $N(x) < N(x^2)$ and so $\rho^{-1}(N(x)) < \rho^{-1}(N(\rho(x))) = N^\rho(x)$. But, once that $x < \sqrt{x}$ for each $x \in (0,1)$, we have that $N(x) < N^\rho(x)$ for each $x \in (0,1)$. So, $N < N^\rho$. The proof that $N^{\rho^{-1}} < N$ is analogous. ■

Corollary 2.1 *There exists neither a lesser nor a greater strict (strong) fuzzy negation.*

Proof: Straightforward from propositions 2.2 and 2.3. ■

The following theorem stated by Enric Trillas in [51], presents a strong relation between automorphism and strong fuzzy negations.

Proposition 2.4 *A function $N : U \longrightarrow U$ is a strong fuzzy negation if and only if there exists an automorphism ρ such that $N = C^\rho$, where C is the strong fuzzy negation $C(x) = 1 - x$.*

Proof: See [51]. ■

This theorem was generalized by János Fodor in [19] for strict fuzzy negations.

Proposition 2.5 *A function $N : U \longrightarrow U$ is a strict fuzzy negation if and only if there exist automorphisms ρ_1 and ρ_2 such that $N = \rho_1 \circ C \circ \rho_2$, where C is the strong fuzzy negation $C(x) = 1 - x$.*

Proof: See [19]. ■

Mirko Navara, in order to answer a question stated by himself in [39], introduced in [40] the notion of **negation-preserving automorphism** as being an automorphism which comutes with the usual negation $C(x) = 1 - x$, i.e. an automorphism ρ such that $\rho(C(x)) = C(\rho(x))$. Here it is introduced a natural generalization of this notion.

Let N be a fuzzy negation. An automorphism ρ is **N-preserving automor-**

phism if for each $x \in U$,

$$\rho(N(x)) = N(\rho(x)). \tag{3}$$

The next proposition is a generalization of [40, Proposition 4.2].

Proposition 2.6 *Let N be a strong fuzzy negation and ρ be an automorphism on $[0, e]$, i.e. a continuous increasing function such that $\rho(0) = 0$ and $\rho(e) = e$, where e is the unique equilibrium point of N. Then $\rho^N : U \to U$, defined by*

$$\rho^N(x) = \begin{cases} \rho(x) & \text{if } x \leq e \\ N(\rho(N(x))) & \text{if } x > e, \end{cases} \tag{4}$$

is an N-preserving automorphism. All N-preserving automorphisms are of this form.

Proof: If $x < e$ then by $N4$, $e = N(e) < N(x)$ and so

$$\begin{aligned} \rho^N(N(x)) &= N(\rho(N(N(x)))) && \text{because } N(x) > e \\ &= N(\rho(x)) && \text{because } N \text{ is strong} \\ &= N(\rho^N(x)) && \text{because } x \leq e. \end{aligned}$$

If $x > e$ then by $N4$, $N(x) < e$ and so

$$\begin{aligned} \rho^N(N(x)) &= \rho(N(x)) && \text{because } N(x) < e \\ &= N(N(\rho(N(x)))) && \text{because } N \text{ is strong} \\ &= N(\rho^N(x)) && \text{because } x > e. \end{aligned}$$

If $x = e$ then, trivially, $\rho^N(N(x)) = e = N(\rho^N(x))$.

On the other hand, if ρ' is an N-preserving automorphism then $\rho : [0, e] \to [0, e]$ defined by $\rho(x) = \rho'(x)$ is such that $\rho(e) = \rho'(N(e)) = N(\rho'(e)) = N(\rho(e))$ and so $\rho(e) = e$, the other properties that show that ρ is an automorphism on $[0, e]$ are inherited from ρ' which is an automorphism. Thus, if $x \leq e$ then $\rho'(x) = \rho^N(x)$. If $x > e$ then

$$\begin{aligned} \rho'(x) &= \rho'(N(N(x))) && \text{because } N \text{ is strong} \\ &= N(\rho'(N(x))) && \text{by equation (3)} \\ &= \rho^N(x) && \text{by equation (4)} \end{aligned}$$

Therefore, $\rho' = \rho^N$, i.e. all N-preserving automorphism have the form of equation (4). ■

Proposition 2.7 *Let N be a strong fuzzy negation and ρ be an automorphism on $[0, e]$, where e is the equilibrium point of N. Then $\rho^{N^{-1}}$ is an N-preserving automorphism.*

Proof: By Proposition 2.6 ρ^N is an N-preserving automorphism. Let $x \in U$.

$$\begin{aligned} \rho^{N^{-1}}(N(x)) &= \rho^{N^{-1}}(N(\rho^N(\rho^{N^{-1}}((x))))) \\ &= \rho^{N^{-1}}(\rho^N(N(\rho^{N^{-1}}((x))))) \text{ by equation (3)} \\ &= N(\rho^{N^{-1}}((x))) \end{aligned}$$

So by equation (3), $\rho^{N^{-1}}$ is also an N-preserving automorphism. ■

3 Best Interval Representations

Let $\mathbb{U}$ be the set of subintervals of U, i.e. $\mathbb{U} = \{[a, b]/0 \leq a \leq b \leq 1\}$. The interval set has two projections $l : \mathbb{U} \longrightarrow U$ and $r : \mathbb{U} \longrightarrow U$ defined by:

$$l([a, b]) = a \text{ and } r([a, b]) = b.$$

As convention, for each $X \in \mathbb{U}$, $l(X)$ and $r(X)$ will also be denoted by $\underline{X}$ and $\overline{X}$, respectively.

Some natural partial orders can be defined on $\mathbb{U}$ [12]. The most used in the context of interval mathematics and which we consider in this work, are the following.

(1) Product:

$$X \leq Y \text{ if and only if } \underline{X} \leq \underline{Y} \text{ and } \overline{X} \leq \overline{Y}$$

(2) Inclusion order:

$$X \subseteq Y \text{ if and only if } \underline{X} \geq \underline{Y} \text{ and } \overline{X} \leq \overline{Y}$$

For each interval $X \in \mathbb{U}$, these orders determine four sets, which form, up to least of the boundary, a partition of $\mathbb{U}$.

- $\uparrow X = \{Y/X \leq Y\}$
- $\downarrow X = \{Y/Y \leq X\}$

- $\Uparrow X = \{Y/X \subseteq Y\}$
- $\Downarrow X = \{Y/Y \subseteq X\}$

These, partition is illustrated in Figure 1.

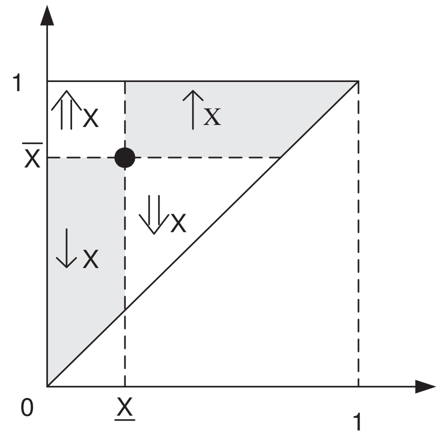

Fig. 1. Partition of $\mathbb{U}$.

A function $F : \mathbb{U} \longrightarrow \mathbb{U}$ is an **interval representation** of a function $f : U \longrightarrow U$ if, for each $X \in \mathbb{U}$ and $x \in X$, $f(x) \in F(X)$.

Notice that this notion coincides with the notions of "inclusion function for f" in [26, equation 2.83] and "interval extension of f" in [27, Definition 1.3]. Nevertheless, the idea of [46] to use a new name was to remark that intervals can be seen as representations of the real numbers belonged to the interval and associated with the **correctness of interval computations** pointed out by Hicker in [24].

An interval representation $F : \mathbb{U} \longrightarrow \mathbb{U}$ of a function $f : U \longrightarrow U$ is a **better representation** of another interval representation $G : \mathbb{U} \longrightarrow \mathbb{U}$ of f, denoted by $G \sqsubseteq F$, if for each $X \in \mathbb{U}$, $F(X) \subseteq G(X)$. Thus, for each real function, there is a natural partial order between its interval representations.

Notice that, the notion of range of a real function applied to intervals; $f([a, b])$, could be seen as an operator which maps those functions on interval functions. Nevertheless, sometimes $f([a, b])$ is not an interval and therefore it is not a valid object in Moore arithmetic (i.e. is not a total interval operation), which is a fundamental requirement for interval operations [24]. Thus, in order to obtain an operator which transforms real functions into interval functions, the range is not a suitable operator. The next definition, introduced in [46], overcomes this problem, by taking the hull interval* of the range of f.

* The hull interval of a set $X \subseteq \mathbb{R}$ is the narrowest interval containing X [26].

For each real function $f : U \longrightarrow U$, the interval function $\widehat{f} : \mathbb{U} \longrightarrow \mathbb{U}$ defined by

$$\widehat{f}(X) = [\inf\{f(x)/x \in X\}, \sup\{f(x)/x \in X\}]$$

is called **the canonical interval representation of** f [46].

Notice that $\widehat{f}$ is well defined and therefore satisfies the totality Hickey requirement for interval operations [24]. Moreover, $\widehat{f}$ is an interval representation of f and for any other interval representation F of f, $F \sqsubseteq \widehat{f}$. In other words, the interval function $\widehat{f}$ returns a narrower interval than any other interval representation of f; i.e. $\widehat{f}$ is the optimal or the **best interval representation** of f; see Hickey et al. [24].

Both, range and best interval representations, i.e. $f(X)$ and $\widehat{f}(X)$, coincide just when f is continuous, i.e. if f is continuous, then for each $X \in \mathbb{U}$, $\widehat{f}(X) = \{f(x)/x \in X\} = f(X)$.

An interval function $F : \mathbb{U} \longrightarrow \mathbb{U}$ **preserves degenerate intervals** if for each $x \in U$, $F([x, x])$ is a degenerate interval[⋆⋆].

An interval function $F : \mathbb{U} \rightarrow \mathbb{U}$ is **representable**[⋆⋆⋆] if there exist functions $f_1, f_2 : U \rightarrow U$ such that, for each $X \in \mathbb{U}$, it holds that $F(X) = [f_1(p_1(X)), f_2(p_2(X))]$, where $p_1, p_2 \in \{l, r\}$.

4 Quasi-metrics and continuity

A **quasi-metric** over a set A is a function $d : A \times A \rightarrow \mathbb{R}$, such that

(a) $d(a, a) = 0$,
(b) $d(a, c) \leq d(a, b) + d(b, c)$ and
(c) $d(a, b) = d(b, a) = 0 \Rightarrow a = b$

A **quasi-metric space** is a pair (A, q), where A is a set and q a quasi-metric over A. For every quasi-metric q, it is always possible to define another quasi-metric, called **conjugated quasi-metric**, defined by $\overline{q}(a, b) = q(b, a)$ [48].

A quasi-metric d is a **metric** if it also satisfies (d) $d(a, b) = d(b, a)$ for each $a, b \in A$. Clearly (d) implies (c). For every quasi-metric q it is possible to define a metric $q^* : A \times A \rightarrow \mathbb{R}$ as follows: $q^*(a, b) = max\{q(a, b), \overline{q}(a, b)\}$.

⋆⋆ Intervals of the form $[x, x]$ are called degenerate intervals.
⋆⋆⋆ See [14], for representable t-norms.

An example of a metric on U is the usual distance for real numbers, $d(x, y) = |x - y|$.

An interval can be seen as a set of real numbers, as a kind of number and as an information of a real number. Each of these notions imply in a classification for intervals and therefore determine a criteria of proximity. When intervals are seen as a kind of number, the associated distance is the metric introduced by Ramon Moore in [38].

Given two intervals $X, Y \in \mathbb{IR}$, the **distance of Moore** between X and Y is defined by $d_M(X, Y) = max(|\underline{Y} - \underline{X}|, |\overline{X} - \overline{Y}|)$

When intervals are seen as an information about a real number, the criteria of proximity is established using the quasi-metric introduced by Benedito Aciòly and Benjamín Bedregal in [1].

Given two intervals $X, Y \in \mathbb{IR}$, the **Aciòly-Bedregal quasi-metric** between X and Y is defined by $q_S(X, Y) = max(\underline{Y} - \underline{X}, \overline{X} - \overline{Y}, 0)$

Notice that $q_S^* = d_M$.

Given two real numbers $x, y \in \mathbb{R}$, the **right quasi-metric** between x and y is defined by $q_r(x, y) = max(x - y, 0)$. The conjugated of q_r is denoted by q_l, the **left quasi-metric**.

Notice that $q_r^* = q_l^* = d$.

A function $f : A \to B$, where (A, q) and (B, q') are quasi-metric spaces, is called (q, q')-**continuous at** $a \in A$ if, for every $\epsilon > 0$, there is $\delta > 0$, such that for every $x \in A$, if $q(x, a) < \delta$, then $q'(f(x), f(a)) < \epsilon$. f is a (q, q')-**continuous function**, if it is continuous in every $a \in A$. When q and q' are clear from the context it will be omitted.

Example 4.1 *The function $deg : \mathbb{R} \to \mathbb{IR}$ defined by $deg(x) = [x, x]$ clearly is (d, d_M)-continuous and (d, q_S)-continuous. In fact, $d(x, y) = d_M(deg(x), deg(y)) = q_S(deg(x), deg(y))$, so it is sufficient to consider $\delta = \epsilon$.*

Composition and Cartesian product preserve continuity:

Proposition 4.1 *Let (A_1, q_1), (A_2, q_2), (A_3, q_3) and (A_4, q_4) be quasi-metric spaces. Then*

(1) For each $i, j \in \{1, 2, 3, 4\}$, $(A_i \times A_j, q_i \times q_j)$ is a quasi-metric space where $q_i \times q_j : (A_i \times A_j) \times (A_i \times A_j) \longrightarrow \mathbb{R}$ defined by $q_i \times q_j((x_1, x_2), (y_1, y_2)) = \sqrt{q_i(x_1, y_1)^2 + q_j(x_2, y_2)^2}$.

(2) $f : A_1 \to A_2$, $g : A_2 \to A_3$ and $h : A_3 \longrightarrow A_4$ are (q_1, q_2), (q_2, q_3)

and (q_3, q_4)*-continuous, respectively, if and only if,* $g \circ f : A_1 \rightarrow A_3$ *and* $f \times h : A_1 \times A_3 \longrightarrow A_2 \times A_4$ *are* (q_1, q_3) *and* $(q_1 \times q_3, q_2 \times q_4)$*-continuous, respectively.*

Proof: It is a natural extension of well known properties in metric spaces. ■

Functions which are (d_M, d_M)-continuous are said Moore continuous and functions which are (q_S, q_S)-continuous are said Scott continuous, because this notion of continuity coincides with the continuity based on Domain Theory introduced by Dana Scott for the continuous domain $(\mathbb{IR}, \supseteq)$ (for more information on this subject see [47,1,17,46]).

The relation between the continuity on real numbers and the above continuities, adapted to our context. i.e. for U instead of $\mathbb{R}$, are stated in the following theorem and proposition:

Theorem 4.1 *Let* $f, g : U \longrightarrow U$ *be antitonic functions such that* $f \leq g$*. The following statements are equivalent:*

(1) f *and* g *are continuous;*
(2) $\mathbb{I}_{[f,g]}$ *is Scott continuous;*
(3) $\mathbb{I}_{[f,g]}$ *is Moore continuous.*

where

$$\mathbb{I}_{[f,g]}(X) = [f(\overline{X}), g(\underline{X})]. \tag{5}$$

Proof: Clearly the projections l and r are (q_S, q_l) and (q_S, q_r)-continuous, respectively. On the other hand, the functions $\delta : \mathbb{U} \longrightarrow \mathbb{U} \times \mathbb{U}$ and $i : U \times U \longrightarrow \mathbb{U}$ defined by $\delta(X) = (X, X)$ and $i(x, y) = [\min(x, y), \max(x, y)]$ are $(q_S, q_S \times q_S)$ and $(d \times d, q_S)$-continuous, respectively. Since, $\mathbb{I}_{[f,g]} = i \circ (f \times g) \circ (r \times l) \circ \delta$, then by Proposition 4.1, $\mathbb{I}_{[f,g]}$ is Scott continuous if and only if f and g are continuous.

Analogously to the previous case, it is possible to prove that $\mathbb{I}_{[f,g]}$ is Moore continuous if and only if f and g are continuous. ■

Theorem 4.2 *Let* $f, g : U \longrightarrow U$ *be isotonic functions such that* $f \leq g$*. The following statements are equivalent:*

(1) f *and* g *are continuous;*
(2) $I_{[f,g]}$ *is Scott continuous;*
(3) $I_{[f,g]}$ *is Moore continuous.*

where

$$I_{[f,g]}(X) = [f(\underline{X}), g(\overline{X})]. \quad (6)$$

Proof: Analogous to the previous theorem. ■

Proposition 4.2 *Let $f : U \longrightarrow U$ be a function. The following statements are equivalent:*

(1) f is continuous;
(2) $\widehat{f}$ is Scott continuous;
(3) $\widehat{f}$ is Moore continuous.

Proof: See [46, theorems 5.1 and 5.2]. ■

Notice that, in spite of $\mathbb{I}_{[f,f]} = \widehat{f}$ and $I_{[f,f]} = \widehat{f}$, this proposition is not a corollary of theorems 4.1 and 4.2, because that in this proposition does not require f to be monotonic.

5 Interval fuzzy negations

Several ways to extend fuzzy negations and their subclasses of strict and strong fuzzy negation are given in the literature, see for example [23,21,41,13,5,11]. The extension provided by Benjamín Bedregal and Adriana Takahashi in [5], which is adopted here, takes into account the representation aspects of interval constructions and the fact that interval mathematics admits two natural partial order and two continuity notions. Nevertheless in [5,50] it was only made a superficial study of the properties of interval fuzzy implications.

A function $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$ is an **interval fuzzy negation** if $\forall X, Y \in \mathbb{U}$

- $\mathbb{N}$1: $\mathbb{N}([0,0]) = [1,1]$ and $\mathbb{N}([1,1]) = [0,0]$.
- $\mathbb{N}$2a: If $X \leq Y$ then $\mathbb{N}(Y) \leq \mathbb{N}(X)$, and
- $\mathbb{N}$2b: If $X \subseteq Y$ then $\mathbb{N}(X) \subseteq \mathbb{N}(Y)$.

$\mathbb{N}$ is a **strict interval fuzzy negation** if it also satisfies the properties

- $\mathbb{N}$3a: $\mathbb{N}$ is Moore Continuous,
- $\mathbb{N}$3b: $\mathbb{N}$ is Scott Continuous,
- $\mathbb{N}$4a: If $X < Y$ then $\mathbb{N}(Y) < \mathbb{N}(X)$, and
- $\mathbb{N}$4b: If $X \subset Y$ then $\mathbb{N}(X) \subset \mathbb{N}(Y)$.

Theorem 5.1 *Let $N_1 : U \longrightarrow U$ and $N_2 : U \longrightarrow U$ be fuzzy negations such that $N_1 \leq N_2$. Then $\mathbb{I}_{[N_1,N_2]} : \mathbb{U} \rightarrow \mathbb{U}$ defined as in equation (5) is an interval* *fuzzy negation. If N_1 and N_2 are strict then $\mathbb{I}_{[N_1,N_2]}$ is also a strict interval fuzzy negation.*

Proof: Since, $\underline{X} \leq \overline{X}$ then by $N2$ property and because $N_1 \leq N_2$, $N_1(\overline{X}) \leq N_1(\underline{X}) \leq N_2(\underline{X})$. Therefore, $\mathbb{I}_{[N_1,N_2]}(X)$ is well defined. The following items prove that $\mathbb{I}_{[N_1,N_2]}$ satisfies properties $\mathbb{N}1$ to $\mathbb{N}4$b.

- $\mathbb{N}1$: straightforward.
- $\mathbb{N}2$a: If $X \leq Y$ then $\underline{X} \leq \underline{Y}$ and $\overline{X} \leq \overline{Y}$. So, by $N2$ property, $N_1(\overline{Y}) \leq N_1(\overline{X})$ and $N_2(\underline{Y}) \leq N_2(\underline{X})$. Therefore, $\mathbb{I}_{[N_1,N_2]}(Y) = [N_1(\overline{Y}), N_2(\underline{Y})] \leq [N_1(\overline{X}), N_2(\underline{X})] = \mathbb{I}_{[N_1,N_2]}(X)$.
- $\mathbb{N}2$b: If $X \subseteq Y$ then $\underline{Y} \leq \underline{X}$ and $\overline{X} \leq \overline{Y}$ and therefore, by $N2$ property, $N_1(\overline{Y}) \leq N_1(\overline{X})$ and $N_2(\underline{X}) \leq N_2(\underline{Y})$. So, $\mathbb{I}_{[N_1,N_2]}(X) = [N_1(\overline{X}), N_2(\underline{X})] \subseteq [N_1(\overline{Y}), N_2(\underline{Y})] = \mathbb{I}_{[N_1,N_2]}(Y)$.
- $\mathbb{N}3$a and $\mathbb{N}3$b: Follows straightforward from Theorem 4.1.
- $\mathbb{N}4$a: If $X < Y$ then 1) $\underline{X} < \underline{Y}$ and $\overline{X} \leq \overline{Y}$, or 2) $\underline{X} \leq \underline{Y}$ and $\overline{X} < \overline{Y}$. For case 1), by $N2$ and $N4$, $N_1(\underline{Y}) < N_1(\underline{X})$ and $N_1(\overline{Y}) \leq N_1(\overline{X})$, and so, $[N_1(\overline{Y}), N_1(\underline{Y})] < [N_1(\overline{X}), N_1(\underline{X})]$. Therefore, $\mathbb{I}_{[N_1,N_2]}(Y) < \mathbb{I}_{[N_1,N_2]}(X)$. Case 2) is analogous.
- $\mathbb{N}4$b: Analogous to $\mathbb{N}4$a.

■

When $N_1 = N_2$ we will denote $\mathbb{I}_{[N_1]}$ instead of $\mathbb{I}_{[N_1,N_2]}$.

The following theorem guarantees that interval fuzzy negations are representable.

Theorem 5.2 *Let $\mathbb{N}$ be an interval fuzzy negation. Define $\underline{\mathbb{N}} : U \longrightarrow U$ and $\overline{\mathbb{N}} : U \longrightarrow U$ by*

$$\underline{\mathbb{N}}(x) = l(\mathbb{N}([x,x])) \text{ and } \overline{\mathbb{N}}(x) = r(\mathbb{N}([x,x])) \quad (7)$$

Then $\underline{\mathbb{N}}$ and $\overline{\mathbb{N}}$ are fuzzy negations and $\mathbb{N} = \mathbb{I}_{[\underline{\mathbb{N}},\overline{\mathbb{N}}]}$. Moreover, if $\mathbb{N}$ is strict then $\underline{\mathbb{N}}$ and $\overline{\mathbb{N}}$ are also strict.

Proof: First we will prove that $\underline{\mathbb{N}}$ and $\overline{\mathbb{N}}$ are fuzzy negations.

- $N1$: $\underline{\mathbb{N}}(0) = l(\mathbb{N}([0,0])) = l([1,1]) = 1$ and $\underline{\mathbb{N}}(1) = r(\mathbb{N}([1,1])) = r([0,0]) = 0$
- $N2$: if $x \geq y$ then $[x,x] \geq [y,y]$ and therefore $\mathbb{N}([x,x]) \leq \mathbb{N}([y,y])$. So, $\underline{\mathbb{N}}(x) \leq \underline{\mathbb{N}}(y)$.

Analogously, for $\overline{\mathbb{N}}$.

Since clearly $\underline{\mathbb{N}} \leq \overline{\mathbb{N}}$, then $\mathbb{I}_{[\underline{\mathbb{N}},\overline{\mathbb{N}}]}$ is well defined. Thus, it only remains to prove that $\mathbb{N} = \mathbb{I}_{[\underline{\mathbb{N}},\overline{\mathbb{N}}]}$.

Since, $X \leq [\overline{X}, \overline{X}]$, then by $\mathbb{N}$2a, $\mathbb{N}([\overline{X}, \overline{X}]) \leq \mathbb{N}(X)$. So, $l(\mathbb{N}([\overline{X}, \overline{X}])) \leq l(\mathbb{N}(X))$. But, $[\overline{X}, \overline{X}] \subseteq X$ and therefore, by $\mathbb{N}$2b, $\mathbb{N}([\overline{X}, \overline{X}]) \subseteq \mathbb{N}(X)$. So, $l(\mathbb{N}(X)) \leq l(\mathbb{N}([\overline{X}, \overline{X}]))$. Thus, $l(\mathbb{N}([\overline{X}, \overline{X}])) = l(\mathbb{N}(X))$. Analogously, it is possible to prove that, $r(\mathbb{N}([\overline{X}, \overline{X}])) = r(\mathbb{N}(X))$. Thus,

$$\begin{aligned}
\mathbb{I}_{[\underline{\mathbb{N}}, \overline{\mathbb{N}}]}(X) &= [\underline{\mathbb{N}}(\overline{X}), \overline{\mathbb{N}}(\underline{X})] \\
&= [l(\mathbb{N}([\overline{X}, \overline{X}])), r(\mathbb{N}([\underline{X}, \underline{X}]))] \\
&= [l(\mathbb{N}(X)), r(\mathbb{N}(X))] \\
&= \mathbb{N}(X)
\end{aligned}$$

Now, we will prove that if $\mathbb{N}$ is strict then $\underline{\mathbb{N}}$ and $\overline{\mathbb{N}}$ are also strict.

If $\mathbb{N}$ is strict then by $\mathbb{N}$3b, $\mathbb{N}$ is Scott continuous and therefore, by Theorem 4.1, $\underline{\mathbb{N}}$ as well as $\overline{\mathbb{N}}$ are continuous.

Let $x, y \in U$ such that $x < y$. Then $[x, y] < [y, y]$ and therefore, by strictness of $\mathbb{N}$, $\mathbb{N}([y, y]) < \mathbb{N}([x, y])$. But, from equation (5), $l(\mathbb{N}([y, y])) = \underline{\mathbb{N}}(y) = l(\mathbb{N}([x, y]))$ and so $\overline{\mathbb{N}}(y) = r(\mathbb{N}([y, y])) < r(\mathbb{N}([x, y])) = \overline{\mathbb{N}}(x)$. Analogously, considering that $[x, x] < [x, y]$ it is possible to prove that $\underline{\mathbb{N}}(y) < \underline{\mathbb{N}}(x)$. Therefore, $\underline{\mathbb{N}}$ and $\overline{\mathbb{N}}$ are strict. ■

Corollary 5.1 *Let* $\mathbb{N} : \mathbb{U} \longrightarrow \mathbb{U}$. $\mathbb{N}$ *is an interval (strict) fuzzy negation if and only if* $\underline{\mathbb{N}}$ *and* $\overline{\mathbb{N}}$ *are (strict) fuzzy negations.*

Proof: Straightforward from theorems 5.1 and 5.2. ■

Thus, the interval (strict) fuzzy negation set coincides with the set of interval functions which are representable by (strict) fuzzy negations.

5.1 Strong Interval Fuzzy Negations

An interval fuzzy negation $\mathbb{N}$ is **strong** if it also satisfies the **involutive** property, i.e. $\forall X \in \mathbb{U}$

- $\mathbb{N}$5: $\mathbb{N}(\mathbb{N}(X)) = X$.

Notice that an analogous result to Theorem 5.1 for involutive (and therefore for strong) fuzzy negations is not hold. For example, $N_1(x) = \frac{1-x}{1+x}$ and $N_2(x) = 1 - x$ are involutive fuzzy negations such that $N_1 \leq N_2$. But, $\mathbb{I}_{[N_1, N_2]}$ is not involutive. Nevertheless, it does not mean that there are not involutive interval fuzzy negations.

Lemma 5.1 *Let* $f : A \longrightarrow A$ *be a function. Then* f *is involutive if and only* *if* $f = f^{-1}$

Proof: Straightforward from definition of inverse. ■

Lemma 5.2 *Let* $\mathbb{N}$ *be an interval strong fuzzy negation. Then* $\mathbb{N}$ *preserves degenerate intervals.*

Proof: Suppose that for some degenerate interval $[x, x]$, $\mathbb{N}([x, x])$ is not a degenerate interval, i.e. $\mathbb{N}([x, x]) = [a, b]$ for some $a < b$. By involution, $\mathbb{N}(\mathbb{N}([x, x])) = [x, x]$ and so $\mathbb{N}([a, b]) = [x, x]$. Let $c = \frac{b-a}{2}$, then because $[c, c] \subseteq [a, b]$ by $\mathbb{N}$2b, $\mathbb{N}([c, c]) = [x, x]$ and so $\mathbb{N}$ has no inverse which is a contradiction to Lemma 5.1. ■

Proposition 5.1 *Let* $\mathbb{N}$ *be an interval strong fuzzy negation. Then,* $\underline{\mathbb{N}}$ *and* $\overline{\mathbb{N}}$ *are strong fuzzy negations.*

Proof: By Theorem 5.2, $\underline{\mathbb{N}}$ and $\overline{\mathbb{N}}$ are fuzzy negations. Thus, it only remains to prove that both are involutive.

$$\begin{aligned}
\underline{\mathbb{N}}(\underline{\mathbb{N}}(x)) &= \underline{\mathbb{N}}(l(\mathbb{N}([x, x]))) && \text{By equation (7)} \\
&= l(\mathbb{N}([l(\mathbb{N}([x, x])), l(\mathbb{N}([x, x]))])) && \text{By equation (7)} \\
&= l(\mathbb{N}([l(\mathbb{N}([x, x])), r(\mathbb{N}([x, x]))])) && \text{By Lemma 5.2} \\
&= l(\mathbb{N}(\mathbb{N}([x, x]))) && \text{By definition of } l \text{ and } r \\
&= x && \text{Because } \mathbb{N} \text{ is involutive}
\end{aligned}$$

The case of $\overline{\mathbb{N}}$ is analogous. ■

Corollary 5.2 *Let* $\mathbb{N}$ *be an interval strong fuzzy negation. Then,* $\underline{\mathbb{N}^{-1}} = \underline{\mathbb{N}}^{-1}$ *and* $\overline{\mathbb{N}^{-1}} = \overline{\mathbb{N}}^{-1}$

Proof: By Lemma 5.1, $\mathbb{N} = \mathbb{N}^{-1}$ and so $\underline{\mathbb{N}^{-1}} = \underline{\mathbb{N}}$. By Proposition 5.1, $\underline{\mathbb{N}}$ is strong and therefore, by Lemma 5.1, $\underline{\mathbb{N}} = \underline{\mathbb{N}}^{-1}$. So, $\underline{\mathbb{N}^{-1}} = \underline{\mathbb{N}}^{-1}$. ■

Theorem 5.3 *Let* $\mathbb{N}$ *be an interval fuzzy negation. Then,* $\mathbb{N}$ *is strong if and only if there exists a strong fuzzy negation* N*, such that* $\mathbb{N} = \mathbb{I}_{[N]}$.

Proof: ($\Rightarrow$) From Theorem 5.2, $\mathbb{N} = \mathbb{I}_{[\underline{\mathbb{N}}, \overline{\mathbb{N}}]}$. Then, because $\mathbb{N}$ is involutive, for each $x \in U$,

$$
\begin{aligned}
[x,x] &= \mathbb{N}(\mathbb{N}([x,x])) \\
&= \mathbb{I}_{[\underline{\mathbb{N}},\overline{\mathbb{N}}]}(\mathbb{I}_{[\underline{\mathbb{N}},\overline{\mathbb{N}}]}([x,x])) \\
&= \mathbb{I}_{[\underline{\mathbb{N}},\overline{\mathbb{N}}]}([\underline{\mathbb{N}}(x),\overline{\mathbb{N}}(x)]) \\
&= [\underline{\mathbb{N}}(\overline{\mathbb{N}}(x)),\overline{\mathbb{N}}(\underline{\mathbb{N}}(x))]
\end{aligned}
$$

So, $\underline{\mathbb{N}}(\overline{\mathbb{N}}(x)) = x$ and therefore, $\underline{\mathbb{N}}(\underline{\mathbb{N}}(\overline{\mathbb{N}}(x))) = \underline{\mathbb{N}}(x)$. Thus, due to Proposition 5.1, $\underline{\mathbb{N}}$ is involutive, we have that $\overline{\mathbb{N}} = \underline{\mathbb{N}}$.

($\Leftarrow$) On the other hand, from equation (5), for each $X \in \mathbb{U}$, $\mathbb{N}(\mathbb{N}(X)) = \mathbb{I}_{[N]}(\mathbb{I}_{[N]}(X)) = \mathbb{I}_{[N]}([N(\overline{X}), N(\underline{X})]) = [N(N(\underline{X})), N(N(\overline{X}))] = X$. ■

Corollary 5.3 *Let* $\mathbb{N}$ *be a strong interval fuzzy negation. Then* $\mathbb{N}$ *is strict.*

Proof: Straightforward from theorems 5.3 and 5.1, and the fact that strong fuzzy negations are strict. ■

5.2 Interval Fuzzy Negations and Set Operations

Proposition 5.2 *Let* $\mathbb{N}$ *be an interval fuzzy negation and* $X, Y \in \mathbb{U}$*. If* $X \cap Y \neq \emptyset$ *then* $\mathbb{N}(X) \cap \mathbb{N}(Y) = \mathbb{N}(X \cap Y)$

Proof: By Theorem 5.2 $\underline{\mathbb{N}}$ and $\overline{\mathbb{N}}$ are fuzzy negations. Thus, by $N2$,

$$\underline{\mathbb{N}}(\min\{\overline{X},\overline{Y}\}) = \max\{\underline{\mathbb{N}}(\overline{X}),\underline{\mathbb{N}}(\overline{Y})\}$$

and

$$\overline{\mathbb{N}}(\max\{\underline{X},\underline{Y}\}) = \min\{\overline{\mathbb{N}}(\underline{X}),\overline{\mathbb{N}}(\underline{Y})\}.$$

So, $\mathbb{N}(X \cap Y) = \mathbb{N}(X) \cap \mathbb{N}(Y)$. ■

Proposition 5.3 *Let* $\mathbb{N}$ *be an interval fuzzy negation and* $X, Y \in \mathbb{U}$*. Then* $\mathbb{N}(X) \cup \mathbb{N}(Y) = \mathbb{N}(X \cup Y)$*, where* $X \cup Y = [\min\{\underline{X},\underline{Y}\},\max\{\overline{X},\overline{Y}\}]$.

Proof: Analogously to Proposition 5.2. ■

5.3 Best Interval Representation of Fuzzy Negations

As follows, it will be presented a theorem which shows that $\mathbb{I}_{[N]}$ is the best interval representation of N.

Theorem 5.4 *Let* N *be a fuzzy negation. Then*

$$\mathbb{I}_{[N]} = \widehat{N}$$

Proof: If $x \in X$ then $\underline{X} \leq x \leq \overline{X}$ and therefore, by $N2$ property, $N(\overline{X}) \leq N(x) \leq N(\underline{X})$. So, $N(x) \in \mathbb{I}_{[N]}(X)$. Thus, $N(X) \subseteq I[N](X)$. Therefore, because $l(\mathbb{I}_{[N]}(X)) = N(\overline{X})$ and $r(\mathbb{I}_{[N]}(X)) = N(\underline{X})$, $\mathbb{I}_{[N]}(X)$ is the least closed interval containing $N(X)$, i.e. $\mathbb{I}_{[N]} = \widehat{N}$. ■

Clearly, from Corollary 5.1, N is strict if and only if $\widehat{N}$ is strict and from Theorem 5.3, N is strong if and only if $\widehat{N}$ is strong.

The partial order on fuzzy negations can be extended for interval fuzzy negations as follows. Let $\mathbb{N}_1$ and $\mathbb{N}_2$ be interval fuzzy negations, then

$$\mathbb{N}_1 \preceq \mathbb{N}_2 \text{ if and only if for each } X \in \mathbb{U},\ \mathbb{N}_1(X) \leq \mathbb{N}_2(X)$$

Lemma 5.3 *Let N_1 and N_2 be fuzzy negations. If $N_1 \leq N_2$ then $\mathbb{I}_{[N_1]} \preceq \mathbb{I}_{[N_1,N_2]} \preceq \mathbb{I}[N_2]$.*

Proof: Straightforward. ■

Proposition 5.4 *Let $\mathbb{N}$ be an interval fuzzy negation. Then*

$$\widehat{N_\perp} \preceq \mathbb{N} \preceq \widehat{N_\top}.$$

Proof: Straightforward from Theorem 5.2, Lemma 5.3 and equation (1). ■

However, analogously to the punctual case, there are not a lesser and a greater strict and strong interval fuzzy negations.

6 Equilibrium Intervals

Analogously, to fuzzy negations, $E \in \mathbb{U}$ is an **equilibrium interval** for an interval fuzzy negation $\mathbb{N}$ if $\mathbb{N}(E) = E$. Trivially, $[0, 1]$ is an equilibrium interval of any interval fuzzy negation. Thus, if an equilibrium interval E is such that $E \neq [0, 1]$ then E is said a **non-trivial equilibrium interval**.

Proposition 6.1 *Let N_1 and N_2 be fuzzy negations such that $N_1 \leq N_2$. If e_1 and e_2 are the equilibrium points of N_1 and N_2, respectively, then for each equilibrium interval E of $\mathbb{I}_{[N_1,N_2]}$, $[e_1, e_2] \subseteq E$.*

Proof: Notice that by Proposition 2.1, $e_1 \leq e_2$ and so $[e_1, e_2]$ is well defined.

Since, $[\underline{E}, \underline{E}] \leq E$ then, by $\mathbb{N}$2a, $E = \mathbb{I}_{[N_1,N_2]}(E) \leq \mathbb{I}_{[N_1,N_2]}([\underline{E}, \underline{E}])$ and so, $\underline{E} \leq N_1(\underline{E})$. Therefore, by Remark 2.2, $\underline{E} \leq e_1$. Analogously, since $E \leq [\overline{E}, \overline{E}]$ then, by $\mathbb{N}$2a, $\mathbb{I}_{[N_1,N_2]}([\overline{E}, \overline{E}]) \leq \mathbb{I}_{[N_1,N_2]}(E) = E$ and so, by Remark 2.2, $N_2(\overline{E}) \leq \overline{E}$. Therefore, $e_2 \leq \overline{E}$. Hence, $[e_1, e_2] \subseteq E$. ■

Notice that, it does not mean that for each pair of fuzzy negations N_1 and N_2 with an equilibrium point, $\mathbb{I}_{[N_1,N_2]}$ necessarily has a non-trivial equilibrium interval.

Example 6.1 *Let $N_1(x) = 1 - x$ and $N_2(x) = 1 - x^2$. Clearly, $N_1 \leq N_2$ and its equilibrium points are 0.5 and $\frac{\sqrt{5}-1}{2}$, respectively. However, if E is an equilibrium interval for $\mathbb{I}_{[N_1,N_2]}$, then $N_1(\overline{E}) = \underline{E}$ and $N_2(\underline{E}) = \overline{E}$. So, $N_2 \circ N_1(\overline{E}) = \overline{E}$ and $N_1 \circ N_2(\underline{E}) = \underline{E}$. Therefore,*

$$\underline{E} = N_1 \circ N_2(\underline{E}) = N_1(1 - \underline{E}^2) = 1 - (1 - \underline{E}^2) = \underline{E}^2.$$

So, $\underline{E} = 0$ or $\underline{E} = 1$.

Analogously,

$$\overline{E} = N_2 \circ N_1(\overline{E}) = N_2(1 - \overline{E}) = 1 - (1 - \overline{E})^2 = 1 - (1 - 2\overline{E} + \overline{E}^2) = 2\overline{E} - \overline{E}^2.$$

So, $\overline{E}^2 = \overline{E}$ and therefore $\overline{E} = 0$ or $\overline{E} = 1$. Hence, $\mathbb{I}_{[N_1,N_2]}$ has no non-trivial equilibrium interval.

On the other hand, there are interval fuzzy negations with infinite equilibrium intervals. For example, let N be the fuzzy negation $N(x) = 1 - x$, then for each $x \in U$, $E = [\min\{x, 1 - x\}, \max\{x, 1 - x\}]$ is an equilibrium interval of $\widehat{N}$.

Lemma 6.1 *If E_1 and E_2 are equilibrium intervals of an interval fuzzy negation $\mathbb{N}$, then $E_1 \subseteq E_2$ or vice-versa.*

Proof: By Theorem 5.2, $\mathbb{N} = \mathbb{I}_{[\underline{\mathbb{N}},\overline{\mathbb{N}}]}$ and so, $E_1 = \mathbb{N}(E_1) = [\underline{\mathbb{N}}(\overline{E_1}), \overline{\mathbb{N}}(\underline{E_1})]$. Therefore, $\underline{\mathbb{N}}(\overline{E_1}) = \underline{E_1}$ and $\overline{\mathbb{N}}(\underline{E_1}) = \overline{E_1}$. Analogously, $\underline{\mathbb{N}}(\overline{E_2}) = \underline{E_2}$ and $\overline{\mathbb{N}}(\underline{E_2}) = \overline{E_2}$. Thus, if $\underline{E_1} \leq \underline{E_2}$ then, by $N2$, $\overline{\mathbb{N}}(\underline{E_2}) \leq \overline{\mathbb{N}}(\underline{E_1})$ and so $\overline{E_2} \leq \overline{E_1}$. Hence, $E_2 \subseteq E_1$. Analogously, if $\underline{E_2} \leq \underline{E_1}$ then it is possible to prove that $E_1 \subseteq E_2$. ■

Theorem 6.1 *Let $\mathbb{N}$ be an interval fuzzy negation. Then there exists an equilibrium interval E of $\mathbb{N}$ such that for any other equilibrium interval E' of $\mathbb{N}$, we have that $E \subseteq E'$.*

Proof: Let Δ be the set of all equilibrium intervals of $\mathbb{N}$. By Lemma 6.1, $\bigcap_{E\in\Delta} E = [\sup\{\underline{E}/E \in \Delta\}, \inf\{\overline{E}/E \in \Delta\}]$. Since, $[0,1] \in \Delta$ then, $\bigcap_{E\in\Delta} E \neq \emptyset$. So,

$$\begin{aligned}
\mathbb{N}(\textstyle\bigcap_{E\in\Delta} E) &= \mathbb{N}([\sup\{\underline{E}/E \in \Delta\}, \inf\{\overline{E}/E \in \Delta\}]) \\
&= [\underline{\mathbb{N}}(\inf\{\overline{E}/E \in \Delta\}), \overline{\mathbb{N}}(\sup\{\underline{E}/E \in \Delta\})] \\
&= [\sup\{\underline{\mathbb{N}}(\overline{E})/E \in \Delta\}, \inf\{\overline{\mathbb{N}}(\underline{E})/E \in \Delta\}] \\
&= \textstyle\bigcap_{E\in\Delta} \mathbb{N}(E) \\
&= \textstyle\bigcap_{E\in\Delta} E.
\end{aligned}$$

■

Thus, this theorem states that in spite of some interval fuzzy negations admit an infinite quantity of equilibrium intervals, there exists an equilibrium interval which is the narrowest.

Proposition 6.2 *Let N_1 and N_2 be fuzzy negations such that $N_1 \leq N_2$. Then, e is an equilibrium point of N_1 and N_2 if and only if $[e,e]$ is an equilibrium interval of $\mathbb{I}_{[N_1,N_2]}$.*

Proof: Straightforward. ■

Corollary 6.1 *Let N be a fuzzy negation. Then N has an equilibrium point if and only if $\widehat{N}$ has a degenerate equilibrium interval.*

Proof: Straightforward from Theorem 5.4 and Proposition 6.2. ■

Proposition 6.3 *Let $\mathbb{N}$ be an interval fuzzy negation and E be an equilibrium interval of $\mathbb{N}$. Then*

(1) $\mathbb{N}(\downarrow E) \subseteq \uparrow E$,
(2) $\mathbb{N}(\uparrow E) \subseteq \downarrow E$,
(3) $\mathbb{N}(\Downarrow E) \subseteq \Downarrow E$, *and*
(4) $\mathbb{N}(\Uparrow E) \subseteq \Uparrow E$.

Proof:

(1) If $X \leq E$ then by $\mathbb{N}$2a, $E \leq \mathbb{N}(X)$ and so $\mathbb{N}(X) \in \uparrow E$.
(2) If $E \leq X$ then by $\mathbb{N}$2a, $\mathbb{N}(X) \leq E$ and so $\mathbb{N}(X) \in \downarrow E$.
(3) If $X \subseteq E$ then by $\mathbb{N}$2b, $\mathbb{N}(X) \subseteq E$ and so $\mathbb{N}(X) \in \Downarrow E$.
(4) If $E \subseteq X$ then by $\mathbb{N}$2b, $E \subseteq \mathbb{N}(X)$ and so $\mathbb{N}(X) \in \Uparrow E$.

■

Corollary 6.2 *Let $\mathbb{N}$ be an interval fuzzy negation and E be an equilibrium* *interval of* $\mathbb{N}$. *Then*

(1) If $X \leq E$ *then* $X \leq E \leq \mathbb{N}(X)$*,*
(2) If $E \leq X$ *then* $\mathbb{N}(X) \leq E \leq X$*,*
(3) If $X \subseteq E$ *then* $X \subseteq E \subseteq \mathbb{N}(X)$*, and*
(4) If $E \subseteq X$ *then* $\mathbb{N}(X) \subseteq E \subseteq X$*.*

Proof: Straightforward from Proposition 6.3. ■

Nevertheless, interval fuzzy negations do not have an analogous property to Remark 2.2. For example, take into account the interval fuzzy negation $\mathbb{I}_{[N_1,N_2]}$ where N_1 and N_2 are the fuzzy negations defined by equations (8) and (9).

$$N_1(x) = \begin{cases} 1 - \frac{3x}{4} & \text{if } x \leq 0.8 \\ 2(1-x) & \text{if } x > 0.8. \end{cases} \tag{8}$$

$$N_2(x) = \begin{cases} 1 - \frac{x}{2} & \text{if } x \leq 0.8 \\ 3(1-x) & \text{if } x > 0.8. \end{cases} \tag{9}$$

Clearly, the narrowest equilibrium interval of $\mathbb{I}_{[N_1,N_2]}$ is $E = [0.4, 0.8]$ and $\mathbb{I}_{[N_1,N_2]}([0.6, 0.7]) = [0.475, 0.7]$. Thus, $\mathbb{I}_{[N_1,N_2]}([0.6, 0.7]) \leq [0.6, 0.7]$, however $E \not\leq [0.6, 0.7]$.

Moreover, Proposition 2.1 does not have either an analogous to interval fuzzy negation. For example, consider the interval fuzzy negation $\mathbb{I}_{[N_3,N_4]}$ where N_3 and N_4 are the fuzzy negations defined by equations (10) and (11).

$$N_3(x) = \begin{cases} 1 - \frac{12x}{31} & \text{if } x \leq 0.775 \\ \frac{28(1-x)}{9} & \text{if } x > 0.775. \end{cases} \tag{10}$$

$$N_4(x) = \begin{cases} 1 - \frac{9x}{28} & \text{if } x \leq 0.775 \\ \frac{10(1-x)}{3} & \text{if } x > 0.775. \end{cases} \tag{11}$$

Figure 2 shows the relation among the fuzzy negations $N_1, \ldots, N_4$. From that figure it is clear that $\mathbb{I}_{[N_1,N_2]} \preceq \mathbb{I}_{[N_3,N_4]}$ and from equations (10) and (11), we have that the narrowest equilibrium interval of $\mathbb{I}_{[N3,N4]}$ is $E' = [0.7, 0.775]$ and so, $E \not\leq E'$, in fact $E' \subset E$.

Nevertheless, the next proposition presents a weaker interval version of Proposition 2.1. Notice that the punctual version of this proposition is equivalent to

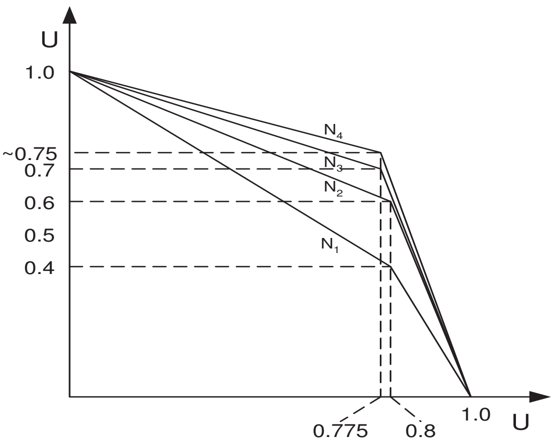

Fig. 2. Comparative shape of four fuzzy negations.

Proposition 2.1.

Proposition 6.4 *Let* $\mathbb{N}_1$ *and* $\mathbb{N}_2$ *be interval fuzzy negations such that* $\mathbb{N}_1 \preceq \mathbb{N}_2$*. If* E_1 *and* E_2 *are, respectively, equilibrium intervals of* $\mathbb{N}_1$ *and* $\mathbb{N}_2$ *then* $E_2 \not< E_1$.

Proof: Since $\mathbb{N}_1 \preceq \mathbb{N}_2$, then $\mathbb{N}_1(E_1) \leq \mathbb{N}_2(E_1)$. But $\mathbb{N}_1(E_1) = E_1$. So,

$$E_1 \leq \mathbb{N}_2(E_1). \tag{12}$$

Suppose that $E_2 < E_1$ then, by $\mathbb{N}$2a, $\mathbb{N}_2(E_1) \leq \mathbb{N}_2(E_2)$. But $\mathbb{N}_2(E_2) = E_2$ and so $\mathbb{N}_2(E_1) \leq E_2$. Therefore, by equation (12), $E_1 \leq E_2$ which is a contradiction. ■

An analogous result to Remark 2.4 is also possible for interval fuzzy negations.

Proposition 6.5 *Let* $E \in \mathbb{U}$*. Then there exists infinitely many interval fuzzy negations having* E *as equilibrium interval.*

Proof: Let $\beta \in [\overline{E}, 1[$. Consider the functions $N_1, N_2 : U \to U$ defined by

$$N_1(x) = \begin{cases} 1 - \frac{(1-\underline{E})x}{\overline{E}} & \text{if } x \leq \overline{E} \\ \frac{(1-x)\underline{E}}{1-\overline{E}} & \text{if } x > \overline{E} \end{cases}$$

$$N_2(x) = \begin{cases} 1 - \frac{(1-\overline{E})x}{\underline{E}} & \text{if } x \leq \beta \\ \frac{(1-x)(1-\frac{(1-\overline{E})\beta}{\underline{E}})}{1-\beta} & \text{if } x > \beta \end{cases}$$

Figure 3, which shows instances of both fuzzy negations, makes clear that N_1 as well as N_2 are strict fuzzy negations and that $N_1 \leq N_2$. Since, clearly,

$N_1(\overline{E}) = \underline{E}$ and $N_2(\underline{E}) = \overline{E}$, then $\mathbb{I}_{[N_1,N_2]}(E) = E$. ■

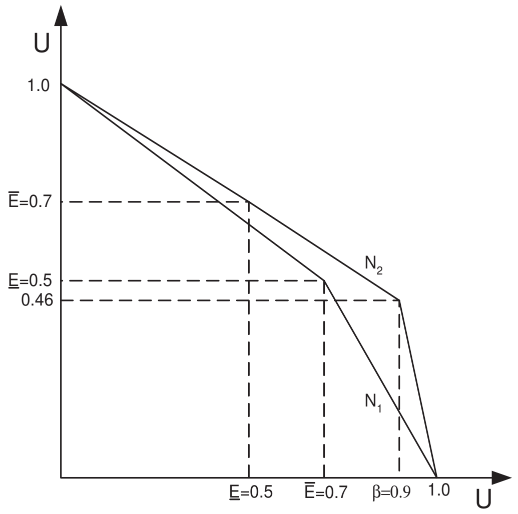

Fig. 3. Example of fuzzy negations, based on fator $\beta = 0.9$, having $E = [0.5, 0.7]$ as equilibrium interval.

6.1 Equilibrium interval of strict interval fuzzy negations

Since, in Example 6.1, N_1 and N_2 are strict, we can conclude that interval strict fuzzy negations can have no non-trivial equilibrium interval.

Proposition 6.6 *Let* $\mathbb{N}$ *be a strict interval fuzzy negation and* e_1 *and* e_2 *the equilibrium points of* $\underline{\mathbb{N}}$ *and* $\overline{\mathbb{N}}$*, respectively. Then* $[e_1, e_2]$ *is an equilibrium interval of* $\mathbb{N}$ *if and only if* $e_1 = e_2$*.*

Proof: ($\Rightarrow$) By Proposition 6.1, $e_1 \leq e_2$. By Theorem 5.2, $\mathbb{N}([e_1, e_2]) = [\underline{\mathbb{N}}(e_2), \overline{\mathbb{N}}(e_1)]$. Thus if $[e_1, e_2]$ is an equilibrium interval of $\mathbb{N}$, then $\underline{\mathbb{N}}(e_2) = e_1$ and $\overline{\mathbb{N}}(e_1) = e_2$. Since, e_1 is the equilibrium point of $\underline{\mathbb{N}}$, then $\underline{\mathbb{N}}(e_2) = \underline{\mathbb{N}}(e_1)$. So, because $\underline{\mathbb{N}}$ is strictly decreasing, $e_1 = e_2$.

($\Leftarrow$) If $e_1 = e_2$ then $\mathbb{N}([e_1, e_2]) = [\underline{\mathbb{N}}(e_2), \overline{\mathbb{N}}(e_1)] = [\underline{\mathbb{N}}(e_1), \overline{\mathbb{N}}(e_2)] = [e_1, e_2]$. ■

Notice that this does not mean that $\underline{\mathbb{N}} = \overline{\mathbb{N}}$.

Example 6.2 *Consider the strict fuzzy negations* $N_1(x) = 1 - x$ *and*

$$N_2(x) = \begin{cases} 1 - 2x^2 & \textit{if } x \leq 0.5 \\ 1 - x & \textit{if } x > 0.5. \end{cases}$$

Clearly $N_1 < N_2$ and both have 0.5 as equilibrium point. Therefore $\mathbb{N} = \mathbb{I}_{[N_1,N_2]}$ has $[0.5, 0.5]$ as an equilibrium interval.

Notice also that not all strict interval fuzzy negations with non-trivial equilibrium interval have a degenerate interval as equilibrium interval. For example, consider the strict fuzzy negations N_1 and N_2 defined in equations (8) and (9). The single non-trivial equilibrium interval of $\mathbb{I}_{[N_1,N_2]}$ is the interval $[0.4, 0.8]$.

6.2 Equilibrium interval of interval strong fuzzy negations

Proposition 6.7 *If $\mathbb{N}$ is an involutive interval fuzzy negation, then $\mathbb{N}$ has a degenerate equilibrium interval.*

Proof: By Theorem 5.3, $\mathbb{N} = \mathbb{I}_{[N]}$ for some involutive fuzzy negations N. By Remark 2.3, there exists an unique equilibrium point for N. Let e be such equilibrium point of N. Then, $[e, e] = [N(e), N(e)] = \mathbb{I}_{[N]}([e, e])$ and so $[e, e]$ is an equilibrium interval for $\mathbb{N}$. ■

The converse of Proposition 6.7 does not hold. For example, the interval fuzzy negation of Example 6.2 is not involutive and has an degenerate interval as equilibrium interval.

Notice that Proposition 6.7 does not imply that the equilibrium interval of an involutive interval fuzzy negation is unique. In fact, as it will be proved as follows, they have an uncountable quantity of equilibrium intervals. For example, for the case of $N(x) = 1 - x$, for each $\epsilon \in [0, 0.5]$, the interval $[0.5 - \epsilon, 0.5 + \epsilon]$ is an equilibrium interval of $\mathbb{I}_{[N]}$.

Theorem 6.2 *If $\mathbb{N}$ is an involutive interval fuzzy negation, then $\mathbb{N}$ has an uncountable quantity of equilibrium intervals.*

Proof: By theorem 5.3, $\mathbb{N} = \mathbb{I}_{[N]}$ for some involutive fuzzy negations N. By Remark 2.3, there exists an unique equilibrium point e for N. Let $\epsilon \in [0, e]$. Then by Remark 2.1, $\epsilon \leq N(\epsilon)$ and so $\mathbb{N}([\epsilon, N(\epsilon)]) = [N(N(\epsilon)), N(\epsilon)] = [\epsilon, N(\epsilon)]$. ■

7 Interval Automorphism

A mapping $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an **interval automorphism** if it is bijective and monotonic w.r.t. the product order [21,22], that is, $X \leq Y$ implies that $\varrho(X) \leq \varrho(Y)$. The set of all interval automorphisms $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is denoted by

$Aut(\mathbb{U})$. Next, it is provided a bijection between the sets $Aut(U)$ and $Aut(\mathbb{U})$. See [21, Theorem 3].

Theorem 7.1 *Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be an interval automorphism. Then there exists an automorphism $\rho : U \longrightarrow U$ such that $\varrho = \mathbb{I}_{[\rho,\rho]}$ as defined in equation (6).*

Proof: See [21, Theorem 2]. ■

For notational simplicity $\mathbb{I}_{[\rho,\rho]}$ will be denoted by $\mathbb{I}_{[\rho]}$.

Interval automorphisms can be generated from a representation of automorphism point of view. In fact, interval automorphisms are the best interval representations of automorphisms.

Theorem 7.2 (Automorphism representation theorem) *Let $\rho : U \to U$ be an automorphism. Then $\widehat{\rho}$ is an interval automorphism.*

Proof: Straightforward from Theorem 7.1 and [4, Theorem 5.2]. ■

Corollary 7.1 *Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$. ϱ is an interval automorphism if and only if there exists an automorphism $\rho : \mathbb{U} \longrightarrow \mathbb{U}$ such that $\varrho = \mathbb{I}_{[\rho]}$.*

Proof: Straightforward from theorems 7.1 and 7.2. ■

Remark 7.1 *As a consequence of this corollary, all the properties of automorphism are preserved for interval automorphism. For example, it can be concluded that, $(Aut(\mathbb{U}), \circ)$ is a group.*

Proposition 7.1 *Let $\rho : U \longrightarrow U$ be an automorphism . Then $\mathbb{I}^{-1}_{[\rho]} = \mathbb{I}_{[\rho^{-1}]}$.*

Proof: Let $X \in \mathbb{U}$.

$$\begin{aligned}
\mathbb{I}_{[\rho]}(\mathbb{I}_{[\rho^{-1}]}(X)) &= \mathbb{I}_{[\rho]}([\rho^{-1}(\underline{X}), \rho^{-1}(\overline{X})]) \\
&= [\rho(\rho^{-1}(\underline{X})), \rho(\rho^{-1}(\overline{X}))] \\
&= X.
\end{aligned}$$

So, $\mathbb{I}_{[\rho^{-1}]} = \mathbb{I}^{-1}_{[\rho]}$. ■

Corollary 7.2 *Let $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ be an interval automorphism. Then ϱ^{-1} is also an interval automorphism.*

Proof: Straightforward from Theorem 7.1 and Proposition 7.1. ■

Notice that fuzzy negations were required by definition to satisfy $\subseteq$-monotonicity. Nevertheless, this property was not required by the definition of interval automorphism. As showed in [4, Corollary 5.1], from the definition of interval automorphism it is possible to prove that interval automorphisms are $\subseteq$-monotonic.

Corollary 7.3 *If ϱ is an interval automorphism then ϱ is inclusion monotonic, that is, if $X \subseteq Y$ then $\varrho(X) \subseteq \varrho(Y)$.*

Analogously, to the alternative definition of automorphism used by [10], there is an alternative characterization for interval automorphisms based on the Moore and Scott continuity. A function $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is **strictly increasing** if, for each $X, Y \in \mathbb{U}$, when $X < Y$ (i.e. $X \leq Y$ and $X \neq Y$) then $\varrho(X) < \varrho(Y)$.

Proposition 7.2 *A function $\varrho : \mathbb{U} \longrightarrow \mathbb{U}$ is an interval automorphism if and only if ϱ is Moore-continuous, strictly increasing, $\varrho([0,0]) = [0,0]$ and $\varrho([1,1]) = [1,1]$.*

Proof: See [4, Proposition 5.1]. ∎

The case of Scott-continuity follows the same setting.

7.1 Interval automorphisms acting on interval fuzzy negations

The following theorems show the properties preserved by interval automorphisms acting on an arbitrary interval fuzzy negation $\mathbb{N}$.

Let $\mathbb{N}$ be an interval fuzzy negation and ϱ be interval automorphism. Then the **action of ϱ on** $\mathbb{N}$ is the function $\mathbb{N}^{\varrho} : \mathbb{U} \longrightarrow \mathbb{U}$ defined by

$$\mathbb{N}^{\varrho}(X) = \varrho^{-1}(\mathbb{N}(\varrho(X))). \tag{13}$$

Theorem 7.3 *Let $\mathbb{N}$ be an interval (strict, strong) fuzzy negation and ϱ be interval automorphism. Then $\mathbb{N}^{\varrho}$ is also an interval fuzzy negation.*

Proof: By Corollary 7.2 ϱ^{-1} is an interval automorphism. So,

- $\mathbb{N}1$: $\mathbb{N}^{\varrho}([0,0]) = \varrho^{-1}(\mathbb{N}(\varrho([0,0]))) = \varrho^{-1}(\mathbb{N}([0,0])) = \varrho^{-1}([1,1]) = [0,0]$. Analogously, it is easy to prove that $\mathbb{N}^{\varrho}([1,1]) = [1,1]$.
- $\mathbb{N}2$a: Let $X, Y \in \mathbb{U}$ such that $X \leq Y$. Then $\varrho(X) \leq \varrho(Y)$. So, $\mathbb{N}(\varrho(Y)) \leq \mathbb{N}(\varrho(X))$ and therefore, $\varrho^{-1}(\mathbb{N}(\varrho(Y))) \leq \varrho^{-1}(\mathbb{N}(\varrho(X)))$, i.e. $\mathbb{N}^{\varrho}(Y) \leq \mathbb{N}^{\varrho}(X)$.
- $\mathbb{N}2$b: Straightforward, because interval automorphisms and interval fuzzy negations are $\subseteq$-monotonic.
- $\mathbb{N}3$a: By Proposition 7.2, ϱ and ϱ^{-1} are Moore-continuous. Thus, if $\mathbb{N}$ satisfies $\mathbb{N}3$a then $\mathbb{N}$ is Moore-continuous and therefore $\mathbb{N}^{\varrho}$ is also Moore-continuous.
- $\mathbb{N}3$b: Analogous to $\mathbb{N}3$a.
- $\mathbb{N}4$: If $X < Y$ then by Proposition 7.2, $\varrho(X) < \varrho(Y)$. So, by $\mathbb{N}4$, $\mathbb{N}(\varrho(X)) < \mathbb{N}(\varrho(Y))$. Therefore, by Proposition 7.2, $\varrho^{-1}(\mathbb{N}(\varrho(X))) < \varrho^{-1}(\mathbb{N}(\varrho(Y)))$, i.e.

$\mathbb{N}^{\varrho}(X) < \mathbb{N}^{\varrho}(Y)$.

■

In the next proposition, it will be proved an analogous result of Proposition 2.3 and Corollary 2.1.

Proposition 7.3 *Let* $\mathbb{N}$ *be a strict (strong) interval fuzzy negation and the interval automorphism* $\varrho(X) = X^2$*, i.e.* $\varrho(X) = [\underline{X}^2, \overline{X}^2]$*. Then,* $\mathbb{N} < \mathbb{N}^{\varrho}$ *and* $\mathbb{N}^{\varrho^{-1}} < \mathbb{N}$.

Proof: Clearly, $\varrho^{-1}(X) = \sqrt{X}$, i.e. $\varrho^{-1}(X) = [\sqrt{\underline{X}}, \sqrt{\overline{X}}]$. Since $X^2 < X$ for each $X \in \mathbb{U} - \{[0,0],[1,1]\}$, then by $\mathbb{N}$4a, $\mathbb{N}(X) < \mathbb{N}(X^2)$ and so $\varrho^{-1}(\mathbb{N}(X)) < \varrho^{-1}(\mathbb{N}(\varrho(X))) = \mathbb{N}^{\varrho}(X)$. But, once $X < \sqrt{X}$ for each $X \in \mathbb{U} - \{[0,0],[1,1]\}$, then $\mathbb{N}(X) < \mathbb{N}^{\varrho}(X)$ for each $X \in \mathbb{U} - \{[0,0],[1,1]\}$. The proof that $\mathbb{N}^{\varrho^{-1}} < \mathbb{N}$ is analogous. ■

Corollary 7.4 *There exists neither a lesser nor a greater strict (strong) interval fuzzy negation.*

Proof: Straightforward from Proposition 7.3. ■

Proposition 7.4 *Let* $\mathbb{N}$ *be an interval fuzzy negation and* ϱ *be an interval automorphism. If* E *is an equilibrium interval of* $\mathbb{N}$ *then* $\varrho^{-1}(E)$ *is an equilibrium interval of* $\mathbb{N}^{\varrho}$.

Proof: $\mathbb{N}^{\varrho}(\varrho^{-1}(E)) = \varrho^{-1}(\mathbb{N}(\varrho(\varrho^{-1}(E)))) = \varrho^{-1}(E)$. ■

Next it will be proved an analogous result for Proposition 2.5.

Theorem 7.4 *A function* $\mathbb{N} : \mathbb{U} \to \mathbb{U}$ *is a strict interval fuzzy negation if and only if there exists interval automorphisms* ϱ_1 *and* ϱ_2 *such that*

$$\mathbb{N}(X) = \varrho_1([1,1] - \varrho_2(X)). \tag{14}$$

Proof: ($\Rightarrow$) Let $\mathbb{N}$ be a strict interval fuzzy negation. Define $\varrho : \mathbb{U} \to \mathbb{U}$ by $\varrho(X) = [1,1] - \mathbb{N}(X)$. Clearly, $\varrho([0,0]) = [0,0]$ and $\varrho([1,1]) = [1,1]$ and, because $\mathbb{N}$ is a strict interval fuzzy negation, ϱ is Moore-continuous and strictly increasing. So, by Proposition 7.2, ϱ is an interval automorphism.

($\Leftarrow$) Suppose that $\mathbb{N}$ is defined by equation (14). Then,

- $\mathbb{N}$1: $\mathbb{N}([0,0]) = \varrho_1([1,1] - \varrho_2([0,0])) = \varrho_1([1,1] - [0,0]) = [1,1]$. Analogously, $\mathbb{N}([1,1]) = \varrho_1([1,1] - \varrho_2([1,1])) = [0,0]$.
- $\mathbb{N}$2a: If $X \leq Y$ then $\varrho_2(X) \leq \varrho_2(Y)$ and so, $[1,1] - \varrho_2(Y) \leq [1,1] - \varrho_2(X)$. Therefore, $\mathbb{N}(Y) = \varrho_1([1,1] - \varrho_2(Y)) \leq \varrho_1([1,1] - \varrho_2(X)) = \mathbb{N}(X)$.

- $\mathbb{N}$2b: If $X \subseteq Y$ then $\varrho_2(X) \subseteq \varrho_2(Y)$ and so, $[1,1] - \varrho_2(X) \subseteq [1,1] - \varrho_2(X)$. Therefore, $\mathbb{N}(Y) = \varrho_1([1,1] - \varrho_2(Y)) \subseteq \varrho_1([1,1] - \varrho_2(X)) = \mathbb{N}(X)$.
- $\mathbb{N}$3a and $\mathbb{N}$3b: Notice that the function $F(X) = [1,1] - X$ is Moore and Scott-continuous and, by Proposition 7.2, ϱ_1 and ϱ_2 are also Moore and Scott-continuous. Thus, because function $\mathbb{N}$ defined by equation (14) is a composition of these functions, then $\mathbb{N}$ is also Moore and Scott-continuous.

Therefore, $\mathbb{N}$ is a strict interval fuzzy negation. ■

In the next proposition, it will be proved an analogous result for Proposition 2.4.

Theorem 7.5 *A function $\mathbb{N} : \mathbb{U} \to \mathbb{U}$ is a strong interval fuzzy negation if and only if there exists an interval automorphism ϱ such that*

$$\mathbb{N}(X) = \varrho^{-1}([1,1] - \varrho(X)). \tag{15}$$

Proof: ($\Rightarrow$) By Theorem 5.3, there exists a strong fuzzy negation N such that $\mathbb{N} = \mathbb{I}_{[N]}$ and by Proposition 2.4 there exist and automorphism ρ such that $N(x) = C^\rho(x)$, i.e. $N(x) = \rho^{-1}(1 - \rho(x))$. So,

$$\begin{aligned}
\mathbb{N}(X) &= \mathbb{I}_{[N]} \\
&= [N(\overline{X}), N(\underline{X})] \\
&= [\rho^{-1}(1 - \rho(\overline{X})), \rho^{-1}(1 - \rho(\underline{X}))] \\
&= \mathbb{I}_{[\rho^{-1}]}([1 - \rho(\overline{X}), 1 - \rho(\underline{X})]) \\
&= \mathbb{I}_{[\rho^{-1}]}([1,1] - [\rho(\underline{X}), \rho(\overline{X})]) \\
&= \mathbb{I}_{[\rho^{-1}]}([1,1] - \mathbb{I}_{[\rho]}(X)) \\
&= \mathbb{I}_{[\rho]}^{-1}([1,1] - \mathbb{I}_{[\rho]}(X))
\end{aligned}$$

($\Leftarrow$) Notice that equation (15) is a particular case of equation (14). Thus, if $\mathbb{N}$ is defined by equation (15), then, straightforward from Corollary 7.2 and Theorem 7.4, $\mathbb{N}$ is an (strict) interval fuzzy negation. So, it only remains to prove $\mathbb{N}$4 property, i.e.:

$$
\begin{aligned}
\mathbb{N}(\mathbb{N}(X)) &= \mathbb{N}(\varrho^{-1}([1,1] - \varrho(X))) \\
&= \varrho^{-1}([1,1] - \varrho(\varrho^{-1}([1,1] - \varrho(X)))) \\
&= \varrho^{-1}([1,1] - ([1,1] - \varrho(X))) \\
&= \varrho^{-1}(\varrho(X)) \\
&= X.
\end{aligned}
$$

Therefore, $\mathbb{N}$ is a strong interval fuzzy negation. ■

7.2 $\mathbb{N}$-preserving interval automorphisms

Let $\mathbb{N}$ be an interval fuzzy negation. An interval automorphism ϱ is **$\mathbb{N}$-preserving interval automorphism** if for each $X \in \mathbb{U}$,

$$\varrho(\mathbb{N}(X)) = \mathbb{N}(\varrho(X)). \tag{16}$$

The next theorem shows that $\mathbb{N}$-preserving interval automorphism is strongly related with the notion of N-preserving automorphism.

Theorem 7.6 *Let ϱ be an interval automorphism, $\mathbb{N}$ be a strong interval fuzzy negation, ρ the automorphism such that $\varrho = \mathbb{I}_{[\rho]}$ (see Theorem 7.1) and N the strong fuzzy negation such that $\mathbb{N} = \mathbb{I}_{[N]}$ (see Theorem 5.3). Then, ϱ is a $\mathbb{N}$-preserving interval automorphism if and only if ρ is a N-preserving automorphism.*

Proof: ($\Rightarrow$) Let $x \in U$, then

$$
\begin{aligned}
\rho(N(x)) &= l(\varrho([N(x), N(x)])) && \text{by Theorem 7.1} \\
&= l(\varrho(\mathbb{N}([x,x]))) && \text{by Theorem 5.3} \\
&= l(\mathbb{N}(\varrho([x,x]))) && \text{by equation (16)} \\
&= l(\mathbb{N}([\rho(x), \rho(x)])) && \text{by Theorem 7.1} \\
&= N(\rho(x)) && \text{by Theorem 5.3}
\end{aligned}
$$

($\Leftarrow$) Let $X \in \mathbb{U}$ then

$$
\begin{aligned}
\varrho(\mathbb{N}(X)) &= \varrho([N(\overline{X}), N(\underline{X})]) && \text{by Theorem 5.3}\\
&= [\rho(N(\overline{X})), \rho(N(\underline{X}))] && \text{by Theorem 7.1}\\
&= [N(\rho(\overline{X})), N(\rho(\underline{X}))] && \text{by equation (3)}\\
&= \mathbb{N}([\rho(\underline{X}), \rho(\overline{X})]) && \text{by Theorem 5.3}\\
&= \mathbb{N}(\varrho(X)) && \text{by Theorem 7.1}
\end{aligned}
$$

■

The next proposition is an interval version of Proposition 2.6 which extends [40, Proposition 4.2].

Proposition 7.5 *Let* $I_E = \{[a,b]/0 \leq a \leq b \leq e\}$*,* $\mathbb{N}$ *be a strong interval fuzzy negation with* $[e,e]$ *as the degenerate equilibrium interval of* $\mathbb{N}$ *and* $\varrho : I_E \to I_E$ *be an interval automorphism, i.e. a bijective and monotonic function. Then* $\varrho^{\mathbb{N}} : \mathbb{U} \to \mathbb{U}$ *defined by*

$$
\varrho^{\mathbb{N}}(X) = \begin{cases} \varrho(X) & \text{if } X \leq [e,e] \\ \mathbb{N}(\varrho(\mathbb{N}(X))) & \text{if } X > [e,e] \\ [\underline{\varrho(X)}, \overline{\mathbb{N}(\varrho(\mathbb{N}(X)))}] & \text{if } \underline{X} < e < \overline{X} \end{cases} \tag{17}
$$

is an $\mathbb{N}$*-preserving interval automorphism. All* $\mathbb{N}$*-preserving interval automorphisms are of this form.*

Proof: By Theorem 7.1, there exists an automorphism ρ (on $[0,e]$) such that for each $X \in I_E$, $\varrho(X) = [\rho(\underline{X}), \rho(\overline{X})]$. Analogously, by Theorem 5.3, there exists a strong fuzzy negation N such that for each $X \in \mathbb{U}$, $\mathbb{N}(X) = [N(\overline{X}), N(\underline{X})]$. Thus,

$$
[\underline{\varrho(X)}, \overline{\mathbb{N}(\varrho(\mathbb{N}(X)))}] = [\rho(\underline{X}), N(\rho(N(\overline{X})))] \tag{18}
$$

If $X < [e,e]$ then by $\mathbb{N}$4a, $[e,e] = \mathbb{N}([e,e]) < \mathbb{N}(X)$ and so

$$
\begin{aligned}
\varrho^{\mathbb{N}}(\mathbb{N}(X)) &= \mathbb{N}(\varrho(\mathbb{N}(\mathbb{N}(X)))) && \text{because } \mathbb{N}(X) > [e,e]\\
&= \mathbb{N}(\varrho(X)) && \text{because } \mathbb{N} \text{ is strong}\\
&= \mathbb{N}(\varrho^{\mathbb{N}}(X)) && \text{because } X \leq [e,e].
\end{aligned}
$$

If $X > [e,e]$ then by $\mathbb{N}$4a, $\mathbb{N}(X) < [e,e]$ and so

$$\begin{aligned}
\varrho^{\mathbb{N}}(\mathbb{N}(X)) &= \varrho(\mathbb{N}(X)) && \text{because } \mathbb{N}(X) < [e,e] \\
&= \mathbb{N}(\mathbb{N}(\varrho(\mathbb{N}(X)))) && \text{because } \mathbb{N} \text{ is strong} \\
&= \mathbb{N}(\varrho^{\mathbb{N}}(X)) && \text{because } X > [e,e].
\end{aligned}$$

If $X = [e,e]$ then, trivially, $\varrho^{\mathbb{N}}(\mathbb{N}(X)) = [e,e] = \mathbb{N}(\varrho^{\mathbb{N}}(X))$.

If $\underline{X} < e < \overline{X}$ then $N(\overline{X}) < N(e) < N(\underline{X})$ and so

$$\begin{aligned}
\varrho^{\mathbb{N}}(\mathbb{N}(X)) &= [\rho(\underline{\mathbb{N}(X)}), N(\rho(N(\overline{\mathbb{N}(X)})))] && \text{by equation (18)} \\
&= [\rho(N(\overline{X})), N(\rho(N(N(\underline{X}))))] && \text{by Theorem 7.1} \\
&= [\rho(N(\overline{X})), N(\rho(\underline{X}))] && \text{because } N \text{ is strong} \\
&= [N(\rho(\overline{X})), N(\rho(\underline{X}))] && \text{by theorem 7.6} \\
&= \mathbb{N}([\rho(\underline{X}), \rho(\overline{X})]) && \text{by Theorem 7.1} \\
&= \mathbb{N}([\rho(\underline{X}), \rho(N(N(\overline{X})))]) && \text{because } N \text{ is strong} \\
&= \mathbb{N}([\rho(\underline{X}), N(\rho(N(\overline{X})))]) && \text{by equation (3)} \\
&= \mathbb{N}(\varrho^{\mathbb{N}}(X)) && \text{by equations (17) and (18)}
\end{aligned}$$

On the other hand, if $\varrho' : \mathbb{U} \to \mathbb{U}$ is a $\mathbb{N}$-preserving interval automorphism then by Theorem 7.6, $\rho' : U \to U$ defined by $\rho'(x) = l(\varrho'([x,x]))$ is a N-preserving automorphism. But, by Proposition 2.6, there exist an automorphism $\rho'' : [0,e] \to [0,e]$ such that $\rho' = \rho''^{N}$. Let $\varrho'' = \mathbb{I}_{[\rho'']}$. Thus, if $X \leq [e,e]$ then

$$\begin{aligned}
\varrho'(X) &= [\rho'(\underline{X}), \rho'(\overline{X})] && \text{by Theorem 7.6} \\
&= [\rho''^{N}(\underline{X}), \rho''^{N}(\overline{X})] && \text{by Proposition 2.6} \\
&= [\rho''(\underline{X}), \rho''(\overline{X})] && \text{by equation (3)} \\
&= \varrho''(X) && \text{by Corollary 7.1} \\
&= \varrho''^{\mathbb{N}}(X) && \text{by equation (17)}
\end{aligned}$$

If $[e,e] < X$ then

$$\begin{aligned}
\varrho'(X) &= [\rho'(\underline{X}), \rho'(\overline{X})] && \text{by Theorem 7.6} \\
&= [\rho''^{N}(\underline{X}), \rho''^{N}(\overline{X})] && \text{by Proposition 2.6} \\
&= [N(\rho''(N(\underline{X}))), N(\rho''(N(\overline{X})))] && \text{by equation (3)} \\
&= \mathbb{N}([\rho''(N(\overline{X})), \rho''(N(\underline{X}))]) && \text{by Theorem 5.3} \\
&= \mathbb{N}(\varrho''([N(\overline{X}), N(\underline{X})])) && \text{by Corollary 7.1} \\
&= \mathbb{N}(\varrho''(\mathbb{N}(X))) && \text{by Theorem 5.3} \\
&= \varrho''^{\mathbb{N}}(X) && \text{by equation (17)}
\end{aligned}$$

If $\underline{X} < e < \overline{X}$ then

$$\begin{aligned}
\varrho'(X) &= [\rho'(\underline{X}), \rho'(\overline{X})] && \text{by Theorem 7.6} \\
&= [\rho''^{N}(\underline{X}), \rho''^{N}(\overline{X})] && \text{by Proposition 2.6} \\
&= [\rho''(\underline{X}), N(\rho''(N(\overline{X})))] && \text{by equation (3)} \\
&= \varrho''^{\mathbb{N}}(X) && \text{by equations (18) and (17)}
\end{aligned}$$

Therefore, $\varrho' = \varrho''^{\mathbb{N}}$, i.e. all $\mathbb{N}$-preserving interval automorphisms have the form of Equation (17). ■

Next, an analogous proposition to Proposition 2.7.

Proposition 7.6 *Let* $\mathbb{N}$ *be a strong interval fuzzy negation. Then* $\varrho^{\mathbb{N}^{-1}}$ *is a* $\mathbb{N}$*-preserving interval automorphism.*

Proof: By Proposition 7.5, $\varrho^{\mathbb{N}}$ is a $\mathbb{N}$-preserving interval automorphism. Let $X \in \mathbb{U}$.

$$\begin{aligned}
\varrho^{\mathbb{N}^{-1}}(\mathbb{N}(X)) &= \varrho^{\mathbb{N}^{-1}}(\mathbb{N}(\varrho^{\mathbb{N}}(\varrho^{\mathbb{N}^{-1}}((X))))) \\
&= \varrho^{\mathbb{N}^{-1}}(\varrho^{\mathbb{N}}(\mathbb{N}(\varrho^{\mathbb{N}^{-1}}((X))))) && \text{by Equation (16)} \\
&= \mathbb{N}(\varrho^{\mathbb{N}^{-1}}(X))
\end{aligned}$$

Therefore, by Equation (16), $\varrho^{\mathbb{N}^{-1}}$ is also a $\mathbb{N}$-preserving interval automorphism. ■

8 Conclusion

In the previous works of the authors [3–5,8,44,7,6,45] it was introduced a generalization for the t-norm, t-conorms, several classes of fuzzy implications and fuzzy negation notions to the set $\mathbb{U}$. These generalizations were made considering the interval representation notion introduced in [46] which is adequate to formalize two fundamental principles of interval computations [24]. 1) *correctness*, where the real output is in the interval output whenever a real input is in an input interval, which is guaranteed by principle of maximum exactness (roundoff "outward", i.e. rounded down and rounded up) and optimal scalar product [2] and 2) *optimality*, where an interval operation is optimal w.r.t. a real operation if the interval result is the narrowest possible containing all possible results of the real operation. In this paper, it was considered the interval generalization for fuzzy negations made in [5] which is also based on the interval representation notion. Notice that this notion is equivalent to the notion of interval valued fuzzy negation in [13,11] which are representable.

The idea in this paper was related to this notion of interval fuzzy negation and its usual subclasses with the interval extension of other concepts that usually is related to fuzzy negations. Thus, it can be noted that most of the usual properties of fuzzy negation are preserved in some sense by these interval extensions.

Others little contributions were made in the context of punctual fuzzy negations, such as the generalization of the concept and properties of N-preserving automorphisms, the preservation of strict and strong fuzzy negation when submitted to the action of an automorphism and their Corollary 2.1.

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