Radicals of the fuzzy Lie algebras

João Carlos da Motta Ferreira and Maria das Graças Bruno Marietto

Universidade Federal do ABC
09210-170, Santo André, Brazil
{joao.cmferreira,graca.marietto}@ufabc.edu.br

Abstract. We apply the concepts of fuzzy sets to lie algebras in order to introduce and to study the notions of solvable and nilpotent fuzzy radicals. We present conditions to prove the existence and uniqueness of such radicals and investigate some of their properties.

Keywords: Fuzzy Lie algebra, Solvable fuzzy radical, Nilpotent fuzzy radical.

1 Introduction

Radical of algebras was a concept that first arose in the classical structure theory of finite-dimensional algebras at the beginning of the 20th century. Initially the radical was taken to be the largest nilpotent ideal of a finite-dimensional associative algebra. Algebras with zero radical (called semi-simple) have obtained a fairly complete description in the classical theory: Any semi-simple finite-dimensional associative algebra is a direct sum of simple matrix algebras over suitable fields. At the same time it turned out that the radical, as well as the largest solvable or largest nilpotent ideal, could be defined in the class of finite-dimensional Lie algebras. Here, as in the associative case, semi-simple Lie algebras turned out to be direct sums of simple algebras of some special form [2].

The notion of fuzzy sets was introduced by L. A. Zadeh [5] and the notions of fuzzy ideals and fuzzy subalgebras of Lie algebras over a field were first introduced by Yehia in [4]. Many mathematicians have been involved in extending the concepts and results of Lie algebras to the fuzzy sets. The aim of this paper is to generalize the concept of solvable and nilpotent radical of Lie algebras to the notion of solvable and nilpotent fuzzy radical of a fuzzy algebra of Lie algebras, respectively, and investigate some of their properties.

2 Fuzzy sets

In this section, we present the basic concepts on fuzzy sets which will be used throughout this work. A notion is introduced and a result is proved for guiding the construction of the main theorems of this work.

Definition 1. *A mapping of a non-empty set $\mathfrak{X}$ into the closed unit interval $[0,1]$ is called a fuzzy set of $\mathfrak{X}$. Let μ be any fuzzy set of $\mathfrak{X}$, then the set $\{\mu(x) \mid x \in \mathfrak{X}\}$ is called the image of μ and is denoted by $\mu(\mathfrak{X})$. The set $\{x \mid x \in \mathfrak{X},\ \mu(x) > 0\}$ is called the support of μ and is denoted by μ^*. In particular, μ is called a finite fuzzy set if μ^* is a finite set, and an infinite fuzzy set otherwise. For all real $t \in [0,1]$ the subset $[\mu]_t = \{x \in \mathfrak{X} \mid \mu(x) \geq t\}$ is called a t-level set of μ.*

A set $S \subset [0,1]$ is said to be an upper well ordered set if for all non-empty subsets $C \subset S$, then $\sup C \in C$ (or $\bigvee C \in C$) [3].

Let $\mathfrak{X}$ be a non-empty set and S an upper well ordered set. One defines the set

$$\mathfrak{F}(\mathfrak{X}, S) = \{\nu \mid \nu \text{ is an arbitrary fuzzy set of } \mathfrak{X} \text{ such that } \nu(\mathfrak{X}) \subseteq S\}.$$

Definition 2. *Let $\mathfrak{X}$ be a non-empty set and $\{\nu_i\}_{i\in I}$ an arbitrary family of fuzzy sets of $\mathfrak{X}$. One defines the fuzzy set of $\mathfrak{X}$ $\bigcup_{i\in I} \nu_i$, called union, as $\big(\bigcup_{i\in I} \nu_i\big)(x) = \bigvee_{i\in I} \nu_i(x)$, for all $x \in \mathfrak{X}$.*

One says that a family of fuzzy sets of $\mathfrak{X}$ $\{\nu_i\}_{i\in I}$ satisfies the second sup property if for all $x \in \mathfrak{X}$ there is an index $i_0 = i_0(x) \in I$ such that $\big(\bigcup_{i\in I} \nu_i\big)(x) = \nu_{i_0}(x)$.

Proposition 1. *Let $\mathfrak{X}$ be a non-empty set and $\{\nu_i\}_{i\in I}$ an arbitrary family of fuzzy sets of $\mathfrak{X}$. Then $\big[\bigcup_{i\in I} \nu_i\big]_t = \bigcup_{i\in I} [\nu_i]_t$ for all $t \in]0,1]$ if, and only if, the family $\{\nu_i\}_{i\in I}$ satisfies the second sup property.*

Proof. The proof of this Proposition can be found in [1].

Corollary 1. *Let $\mathfrak{X}$ be a non-empty set and S an upper well ordered set. Then every family of fuzzy sets $\{\nu_i\}_{i\in I}$ of $\mathfrak{F}(\mathfrak{X}, S)$ satisfies the second sup property.*

In particular, for every family of fuzzy sets $\{\nu_i\}_{i\in I}$ of $\mathfrak{F}(\mathfrak{X}, S)$ we have $\bigcup_{i\in I} \nu_i \in \mathfrak{F}(\mathfrak{X}, S)$.

3 Fuzzy algebras and fuzzy ideals

In this section, we present the basic concepts of fuzzy Lie algebras, and fuzzy Lie subalgebras, fuzzy ideals and solvable (resp., nilpotent) fuzzy ideals, of a fuzzy Lie algebra. Relationships between these concepts with operations of sum and product of the fuzzy sets are studied.

Definition 3. *Let $\mathfrak{L}$ be a Lie algebra over a field $\mathfrak{F}$. A fuzzy set μ of $\mathfrak{L}$ is called a fuzzy Lie algebra of $\mathfrak{L}$ if: (i) $\mu(ax+by) \geq \mu(x) \wedge \mu(y)$, (ii) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ and (iii) $\mu(0) = 1$, for all $a, b \in \mathfrak{F}$ and $x, y \in \mathfrak{L}$.*

A fuzzy set ν of $\mathfrak{L}$ is called a fuzzy subalgebra of μ if ν is a fuzzy Lie algebra of $\mathfrak{L}$ satisfying $\nu(x) \leq \mu(x)$ for all $x \in \mathfrak{L}$.

Clearly, if μ is a fuzzy algebra of $\mathfrak{L}$ then μ^* is a subalgebra of $\mathfrak{L}$. Also, μ is a fuzzy Lie algebra of $\mathfrak{L}$ if, and only if, the t-level sets $[\mu]_t$ are subalgebras of $\mathfrak{L}$, for all $t \in]0,1]$. Moreover, ν is a fuzzy subalgebra of μ, if, and only if, the t-level sets $[\nu]_t$ are subalgebras of $[\mu]_t$, for all $t \in]0,1]$.

Definition 4. *A fuzzy set ν of $\mathfrak{L}$ is called a fuzzy ideal of $\mathfrak{L}$ if: (i) $\nu(ax+by) \geq \nu(x) \wedge \nu(y)$, (ii) $\nu(xy) \geq \nu(x) \vee \nu(y)$ and (iii) $\nu(0) = 1$, for all $a, b \in \mathfrak{F}$ and $x, y \in \mathfrak{L}$.*

A fuzzy set ν of $\mathfrak{L}$ is called a fuzzy ideal of μ if ν is a fuzzy Lie ideal of $\mathfrak{L}$ satisfying $\nu(x) \leq \mu(x)$ for all $x \in \mathfrak{L}$.

Clearly, if ν is a fuzzy ideal of $\mathfrak{L}$ then ν^* is an ideal of $\mathfrak{L}$. Also, ν is a fuzzy ideal of $\mathfrak{L}$ if, and only if, the t-level sets $[\nu]_t$ are ideals of $\mathfrak{L}$, for all $t \in]0,1]$. Moreover, any fuzzy ideal of $\mathfrak{L}$ is a fuzzy algebra of $\mathfrak{L}$ and any fuzzy ideal of μ is a fuzzy subalgebra of μ.

Definition 5. *For any fuzzy Lie algebra μ of $\mathfrak{L}$ the fuzzy set of $\mathfrak{L}$, denoted and defined by $o(x) = 1$ if $x = 0$, and $o(x) = 0$ if $x \neq 0$, is a fuzzy algebra (resp., fuzzy ideal) of μ, called the null fuzzy algebra of μ (resp., null fuzzy ideal of μ). A fuzzy set ν of $\mathfrak{L}$ is the null fuzzy algebra of μ if, and only if, $[\nu]_t = \{0\}$, for all $t \in]0,1]$.*

A fuzzy Lie algebra μ of $\mathfrak{L}$ is called abelian if $\mu^2 = o$ and non abelian otherwise.

Let us observe that, if $S \subset [0,1]$ is an upper well ordered set, then $o \in \mathfrak{F}(\mathfrak{L}, S)$ if, and only if, $0, 1 \in S$. Thus, throughout this work we will always assume that our upper ordered set has the real numbers 0 *and* 1.

Definition 6. *Let $\mathfrak{L}$ be a Lie algebra over a field $\mathfrak{F}$. One defines the following: (i) The fuzzy set $\sum_{i=1}^{n} \nu_i$ of $\mathfrak{L}$ (sum), if $\nu_1, \ldots, \nu_n$ are fuzzy sets of $\mathfrak{L}$, as $\left(\sum_{i=1}^{n} \nu_i\right)(x) = \bigvee \Big\{ \bigwedge_{i=1}^{n} \nu_i(x_i) \mid x = \sum_{i=1}^{n} x_i \Big\}$, for all $x \in \mathfrak{L}$ and (ii) The fuzzy set $\nu_1\nu_2$ of $\mathfrak{L}$ (product), if ν_1 and ν_2 are fuzzy sets of $\mathfrak{L}$, as $(\nu_1\nu_2)(x) = \bigvee \Big\{ \bigwedge_{i=1}^{m} \{\nu_1(c_i) \wedge \nu_2(d_i)\} \mid x = \sum_{i=1}^{m} c_i d_i \Big\}$, for all $x \in \mathfrak{L}$.*

Definition 7. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie algebra of $\mathfrak{L}$. For any fuzzy subalgebra ν of μ we define inductively the derived series of ν as the descending chain of fuzzy subalgebras of μ $\nu^{(1)} \geq \nu^{(2)} \geq \nu^{(3)} \geq \ldots$, by setting $\nu^{(1)} = \nu$ and $\nu^{(n+1)} = \left(\nu^{(n)}\right)^2$ for every $n \geq 1$, and the lower central series of ν as the descending chain of fuzzy algebras of μ $\nu^1 \geq \nu^2 \geq \nu^3 \geq \ldots$, by defying $\nu^1 = \nu$ and $\nu^n = \nu\nu^{n-1}$ for every $n \geq 2$.*

The fuzzy subalgebra ν is said solvable (resp., nilpotent) if there exists an integer $k = k(\nu) \geq 1$ such that $\nu^{(k)} = o$ $\left(\text{resp., } \nu^k = o\right)$. The smallest strict positive integer k such that $\nu^{(k)} = o$ $\left(\text{resp., } \nu^k = o\right)$ is called the index of solvability (resp., index of nilpotency) of ν.

Clearly, the Definition 7 generalizes the concept of solvability (resp., nilpotency) as defined in the class of the Lie algebras [2].

Proposition 2. *Let μ be a fuzzy Lie algebra of $\mathfrak{L}$. If ν_1 and ν_2 are solvable (resp., nilpotent) fuzzy ideals of μ, then $\nu_1 + \nu_2$ and the $\nu_1\nu_2$ are also solvable (resp., nilpotent) fuzzy ideals of μ.*

Proof. The proof of this Proposition can be found in [1].

Corollary 2. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and μ a fuzzy Lie algebra of $\mathfrak{L}$. If S is an upper well ordered set and ν_1 and ν_2 are solvable (resp., nilpotent) fuzzy ideals of μ in $\mathfrak{F}(\mathfrak{L}, S)$, then $\nu_1 + \nu_2$ and $\nu_1\nu_2$ are solvable (resp., nilpotent) fuzzy ideals of μ in $\mathfrak{F}(\mathfrak{L}, S)$.*

4 The solvable and nilpotent fuzzy radical

In this section we present the main result of this work. We prove that every fuzzy Lie algebra has a unique maximal solvable (resp., nilpotent) fuzzy ideal, called the solvable (resp., nilpotent) fuzzy radical. Let us begin by introducing the following definition.

Definition 8. *Let μ be a fuzzy Lie algebra of $\mathfrak{L}$ and S is an upper well ordered set. One says that a fuzzy subalgebra ν of μ is a maximal element of μ in $\mathfrak{F}(\mathfrak{L}, S)$ if $\nu \in \mathfrak{F}(\mathfrak{L}, S)$ and for every fuzzy algebra ν^* of μ in $\mathfrak{F}(\mathfrak{L}, S)$ such that $\nu \subset \nu^*$, then $\nu = \nu^*$. In this case, one says that the fuzzy algebra μ has a maximum in $\mathfrak{F}(\mathfrak{L}, S)$.*

Theorem 1. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and S an upper well ordered set. Then each fuzzy algebra μ of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$ has a maximal solvable (resp., nilpotent) fuzzy ideal in $\mathfrak{F}(\mathfrak{L}, S)$.*

Proof. Let μ be a fuzzy algebra of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$ and let us consider the set

$$\Xi = \{\nu \mid \nu \text{ is a solvable fuzzy ideal of } \mu \text{ in } \mathfrak{F}(\mathfrak{L}, S)\}.$$

Obviously, the set Ξ is nonempty and partially ordered by $\leq$. Let us take a subset $\{\nu_i\}_{i\in I}$ of Ξ totally ordered by $\leq$. Let us show that $\nu = \bigcup_{i\in I} \nu_i$ is an upper bound of $\{\nu_i\}_{i\in I}$ in Ξ. In fact, for every $i \in I$, we have $\nu_i(x) \leq \mu(x)$ for all $x \in \mathfrak{L}$. Hence $\nu(x) = \left(\bigcup_{i\in I} \nu_i\right)(x) = \bigvee_{i\in I} \nu_i(x) \leq \bigvee_{i\in I} \mu(x) \leq \mu(x)$, for all $x \in \mathfrak{L}$. Now let us consider arbitrary elements $i, j \in I$. As the set $\{\nu_i\}_{i\in I}$ is totally ordered by $\leq$, then either $\nu_i \leq \nu_j$ or $\nu_j \leq \nu_i$ implying either $\nu_j(x) \wedge \nu_j(y) \geq \nu_i(x) \wedge \nu_j(y)$ or $\nu_i(x) \wedge \nu_i(y) \geq \nu_i(x) \wedge \nu_j(y)$, for all $x, y \in \mathfrak{L}$. Thus for all $x, y \in \mathfrak{L}$ we have $\nu(x + y) = \left(\bigcup_{i\in I} \nu_i\right)(x + y) = \bigvee_{i\in I} \nu_i(x + y) \geq \bigvee_{i\in I} \left(\nu_i(x) \wedge \nu_i(y)\right) \geq \left(\bigvee_{i\in I} \nu_i(x)\right) \bigwedge \left(\bigvee_{j\in I} \nu_j(y)\right) = \nu(x) \wedge \nu(y)$, Next, for all $a \in \mathfrak{F}$ and $x \in \mathfrak{L}$ we have $\nu(ax) = \left(\bigcup_{i\in I} \nu_i\right)(ax) = \bigvee_{i\in I} \nu_i(ax) \geq \bigvee_{i\in I} \nu_i(x) = \nu(x)$. Also, for all $x, y \in \mathfrak{L}$ we have $\nu(xy) = \left(\bigcup_{i\in I} \nu_i\right)(xy) = \bigvee_{i\in I} \nu_i(xy) \geq \bigvee_{i\in I} \left(\nu_i(x) \vee \nu_i(y)\right) = \left(\bigvee_{i\in I} \nu_i(x)\right) \bigvee \left(\bigvee_{j\in I} \nu_j(y)\right) = \nu(x) \vee \nu(y)$.

Finally, we have $\nu(0) = \left(\bigcup_{i\in I} \nu_i\right)(0) = \bigvee_{i\in I} \nu_i(0) = 1$. So ν is a fuzzy ideal of μ.

Now, let us consider $t \in]0, 1]$. Since $\mathfrak{L}$ is finite dimensional, then $\mathfrak{L}$ has a unique maximal solvable ideal. It follows that each t-level set $[\nu_i]_t$, for $i \in I$, is also a solvable ideal of $\mathfrak{L}$ which implies that $\bigcup_{i\in I} [\nu_i]_t$ is a solvable ideal of $\mathfrak{L}$, because the family $\{\nu_i\}_{i\in I}$ of Ξ is totally ordered by $\leq$. By Proposition 1 and Corollary 1, we have that $\left[\bigcup_{i\in I} \nu_i\right]_t$ is a solvable ideal of $\mathfrak{L}$. Therefore, ν is a solvable fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L}, S)$ and so an upper bound of $\{\nu_i\}_{i \in I}$ in Ξ. From the Zorn's lemma, Ξ possesses at least one maximal element.

Similarly, we prove the nilpotent case.

Theorem 2. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$ and S an upper well ordered set. Then every solvable (resp., nilpotent) fuzzy ideal ν of μ in $\mathfrak{F}(\mathfrak{L}, S)$ is contained in a unique maximal solvable (resp., nilpotent) fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L}, S)$, called solvable (resp., nilpotent) fuzzy radical of μ in $\mathfrak{F}(\mathfrak{L}, S)$ and denoted by $\mathfrak{R}(\mu, S)$ (resp., $\mathfrak{N}(\mu, S)$).*

Proof. Let $\mathfrak{S}$ be a maximal solvable (resp., nilpotent) fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L}, S)$. If ν is a solvable (resp., nilpotent) fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L}, S)$, then $\mathfrak{S} + \nu$ is a solvable (resp., nilpotent) fuzzy ideal of μ in $\mathfrak{F}(\mathfrak{L}, S)$, by Corollary 2, and $\mathfrak{S}(x) = \mathfrak{S}(x) \wedge \nu(0) \leq \bigvee \left\{\mathfrak{S}(c) \wedge \nu(d) \mid x = c + d\right\} = (\mathfrak{S} + \nu)(x)$, for all $x \in \mathfrak{L}$. Thus $\mathfrak{S} \leq \mathfrak{S} + \nu$. Since $\mathfrak{S}$ is maximal, then $\mathfrak{S} + \nu \leq \mathfrak{S}$. So $\mathfrak{S} + \nu = \mathfrak{S}$. Hence $\nu(x) = \mathfrak{S}(0) \wedge \nu(x) \leq \bigvee \left\{\mathfrak{S}(c) \wedge \nu(d) \mid x = c + d\right\} = (\mathfrak{S} + \nu)(x)$, for all $x \in \mathfrak{L}$. So $\nu \leq \mathfrak{S}$. Let $\mathfrak{R}(\mu, S) = \mathfrak{S}$ (resp., $\mathfrak{N}(\mu, S) = \mathfrak{S}$) be.

5 Semisimple fuzzy Lie ideals

In this section we introduce the notion of semisimple fuzzy ideals and establish a relation with the solvable fuzzy radical.

Definition 9. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$, S an upper well ordered set and μ a fuzzy ideal of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$. One says that μ is a semisimple fuzzy ideal in $\mathfrak{F}(\mathfrak{L}, S)$ if: (i) μ is a fuzzy ideal non-abelian, (ii) its solvable fuzzy radical in $\mathfrak{F}(\mathfrak{L}, S)$ is o, that is, $\mathfrak{R}(\mu, S) = o$.*

In the following theorem, its demonstration can be found in [1].

Theorem 3. *Let $\mathfrak{L}$ be a finite dimensional Lie algebra over a field $\mathfrak{F}$, S an upper well ordered set and μ a fuzzy ideal of $\mathfrak{L}$ in $\mathfrak{F}(\mathfrak{L}, S)$ non-abelian. If μ is semisimple in $\mathfrak{F}(\mathfrak{L}, S)$, then μ^* is a non solvable ideal of $\mathfrak{L}$.*

Moreover, μ is semisimple in $\mathfrak{F}(\mathfrak{L}, S)$ if, and only if, μ^* does not contain non trivial solvable ideals of $\mathfrak{L}$.

Conjecture. Semisimple fuzzy Lie ideals are direct sums of some special form of fuzzy Lie ideals?

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