Interpreting Fuzzy Connectives from Quantum Computing - Case Study in Reichenbach Implication Class

Adriano Maron,[1] Lidiane Visintin,[1] Renata H. S. Reiser,[1] and Ana Maria Abeijon[2]

[1] Centro de Desenvolvimento Tecnológico, Universidade Federal de Pelotas
Rua Gomes Carneiro, 1 - Centro - CEP 96010-610, Campus Porto, Pelotas, Brazil
e-mail: {akmaron, lvisintin, reiser}@inf.ufpel.edu.br
[2] Centro Politécnico, Universidade Católica de Pelotas
Rua Fèlix da Cunha, 412 CEP: 96010-000 Centro, Pelotas, Brazil
e-mail: anabeijon@terra.com.br

Abstract. This paper shows that quantum computing can be used to extend the class of fuzzy sets, aiming at taking advantage of properties such as quantum parallelism. The central idea associates the states of a quantum register with membership functions of fuzzy subsets, and the rules for the processes of fuzzyfication are performed by unitary quantum transformations. Besides studying the construction from quantum gates to the logical operators such as negation, the paper also introduces the definition of t-norms and t-conorms based on unitary and controlled quantum gates. Such constructors allow modelling and interpreting union, intersection and difference between fuzzy sets. As the main interest, an interpretation for the Reichenbach implication from quantum computing is obtained. The interpretations are acquired when the measuring operation is performed on the corresponding quantum registers. An evaluation of the corresponding computation is implemented and simulated in the visual programming environment VPE-qGM.

1 Introduction

Fuzzy Logic (FL) and Quantum Computing (QC) are important areas of research aiming to collaborate in the description of uncertainty: the former refers to uncertainty modeling in human being's reasoning, while the latter studies the uncertainty of the real world considering the principles of Quantum Mechanics (QM). So, there are many similarities between these two areas of research, which have been highlighted in several scientific papers [10,20,4,22] and [26].

In this context, the logical structure describing the uncertainty associated with the fuzzy set theory can be modeled by means of quantum transformations and quantum states. Thus, it is possible to model quantum algorithms which represent operations on fuzzy sets (union, intersection, difference, implication), and the membership functions encoding quantum states, possibly overlapping.

The QC predicts that quantum algorithms are, in many scenarios, exponentially faster than their classical analogues, see e.g. [9,28,8] and [23]. Considering this statement, it is feasible to investigate the possibility of representing operations on fuzzy sets from quantum transformations. But frequently, such algorithms can be efficiently performed only on quantum computers, which are under development, not always available, and there is still no support for more complex systems.

So, the simulation of quantum algorithms performed by classical computers enables the development of quantum algorithms, anticipating the knowledge about their behavior when run on a quantum hardware. In this scenario, the environment VPE-qGM (Visual Programming Environment for the Quantum Geometric Machine Model), described in [19] and [18], aims to support modeling and simulation of sequential and distributed quantum algorithms, showing the constructions and the evolution of quantum systems from a set of graphical interfaces.

Our main contribution considers the modeling of quantum algorithms for specifying basic fuzzy operations as union, intersection, difference and implication functions. Such operations are also studied in the visual programming approach for ensuring implementation and simulation on VPE-qGM.

This paper is organized as follows: Section 2 presents the fundamental concepts of fuzzy logic membership functions, fuzzy negation, rules and triangular conorms and norms, fuzzy implications and the difference operator. Section 3 brings the main concepts of quantum computing. In Section 4, the study includes the modeling of fuzzy sets from quantum computing, including some classical concepts such as quantum fuzzy states, fuzzy sets and representation of quantum registers from entangled quantum states. Section 5 presents the operations on fuzzy sets modeled from quantum transformations, considering the fuzzy operations of intersection, union, difference and implication, with the expression for each operation, and related result interpretations. Finally, conclusions and further work are discussed in Section 6.

2 Preliminary on Fuzzy Logic

The non well-defined borders sets called fuzzy sets (FS) were introduced in order to overcome the fact that classical sets present limitations to deal with problems where the transitions from one class to another happen smoothly. The definition, properties and operations of FSs are obtained from the generalization of classical set theory (CST), which is a particular case of fuzzy set theory (FST). In CST, operations over classical sets as union, intersection and complement can be expressed applying the characteristic function, which is defined from a subset A of $\mathcal{X} \neq \emptyset$ to the Boolean set $\{0, 1\}$, assigning to each $x \in \mathcal{X}$ an element of a discrete set $\{0, 1\}$ according to the expression:

$$\lambda_A(x) = \begin{cases} 1, & if\ x \in A, \\ 0, & if\ x \notin A; \end{cases} \tag{1}$$

The FST is based on a generalization of the characteristic function for the interval $[0, 1]$. For the membership function $f_A(x) : \mathcal{X} \to [0, 1]$, the element $x \in \mathcal{X}$ belongs to the subset A with a membership degree, given by $f_A(x)$, such that $0 \leq f_A(x) \leq 1$.

Definition 1. *A* ***fuzzy set A*** *related to a set $\mathcal{X} \neq \emptyset$ is given by the expression:*

$$A = \{(x, f_A(x)) : x \in \mathcal{X}\}. \tag{2}$$

2.1 Fuzzy Connectives

Definition 2. *A function* $N : [0,1] \to [0,1]$ *is a* ***fuzzy negation*** *when the conditions hold:*

N1 $N(0) = 1$ *and* $N(1) = 0$*;*
N2 *If* $x \leq y$ *then* $N(x) \geq N(y)$*, for all* $x, y \in [0,1]$.

Fuzzy negations verifying the involutive property:

N3 $N(N(x)) = x$, for all $x \in [0,1]$,

are called strong fuzzy negations. See, e.g. the standard negation: $N_S(x) = 1 - x$.

Definitions of intersection and union between fuzzy subsets can be obtained by application of aggregate functions. In this paper, we consider triangular norms (t-norms) and triangular conorms (t-conorms) [12].

Definition 3. *A* ***t-(co)norm*** *is a binary operation* $(S)T : [0,1] \times [0,1] \longrightarrow [0,1]$ *such that,* $\forall x, y, z \in [0,1]$*, the following properties hold:*

Commutativity:	***T1***: $T(x,y) = T(y,x)$;	***S1***: $S(x,y) = S(y,x)$;
Associativity:	***T2***: $T(T(x,y),z) = T(x,T(y,z))$;	***S2***: $S(S(x,y),z) = S(x,S(y,z))$;
Monotonicity:	***T3***: *if* $x \leq z$ *then* $T(x,y) \leq T(z,y)$;	***S3***: *if* $x \leq z$ *then* $S(x,y) \leq S(z,y)$
Boundary conditions:	***T4***: $T(x,0) = 0$ *and* $T(x,1) = x$;	***S4***: $S(x,1) = 1$ *and* $S(x,0) = x$

There are many references reporting different definitions of t-norms and t-conorms [13]. Herein, we consider the *Algebraic Produtc* and *Algebraic Sum* respectively given as:

$$T_P(x,y) = x \cdot y; \quad S_P(x,y) = x + y - x \cdot y. \tag{3}$$

Definition 4. *A binary function* $I : [0,1]^2 \to [0,1]$ *is an implication operator (implicator) if the following boundary conditions hold:*

I0: $I(1,1) = I(0,1) = I(0,0) = 1$ *and* $I(1,0) = 0$.

In [7,2], additional properties are considered in order to define a fuzzy implication:

Definition 5. *A* ***fuzzy implication*** $I : [0,1]^2 \to [0,1]$ *is an implicator verifying, for all* $x, y, z \in [0,1]$*, the following conditions:*

I1: *Antitonicity in the first argument: if* $x \leq z$ *then* $I(x,y) \geq I(z,y)$*;*
I2: *Isotonicity in the second argument: if* $y \leq z$ *then* $I(x,y) \leq I(x,z)$*;*
I3: *Falsity dominance in the antecedent:* $I(0,y) = 1$*;*
I4: *Truth dominance in the consequent:* $I(x,1) = 1$.

Among the implication classes with explicit representation by fuzzy connectives (negations and aggregation functions) this work considers the class of (S,N)-implication, extending the classical equivalence $p \to q \Leftrightarrow \neg p \vee q$.

Definition 6. *Let* S *be a t-conorm and* N *be a fuzzy negation. A* (S,N)***-implication*** *is a fuzzy implication* $I_{(S,N)} : [0,1]^2 \to [0,1]$ *defined by:*

$$I_{(S,N)}(x,y) = S(N(x),y), \forall x,y \in [0,1]. \tag{4}$$

If N is an involutive function, Eq. (4) defines an S-**implication**[6].
The Reichenbach implication (I_{RB}) expressed as:

$$I_{RB}(x,y) = 1 - x + x \cdot y, \forall x, y \in [0,1], \quad (5)$$

is an S-implicao, obained by a fuzzy negation $N_S(x) = 1-x$ and a t-conorm $S_P(x,y) = x + y - x \cdot y$, previously presented in Eq. (3(b)), respectively.

2.2 Operations over Fuzzy Sets

In analogous way to the construction of classical sets, consider in the following definitions and examples of operations defined over the fuzzy sets $A, B \subseteq \mathcal{X}$.

Definition 7. *The* ***complement of e****A with respect to $\mathcal{X}$, is a fuzzy set $A' = \{(x, f_{A'}) : x \in \mathcal{X}\}$, with $f_{A'} : \mathcal{X} \to [0,1]$ is given by:*

$$f_{A'}(x) = N_S(f_A(x)) = 1 - f_A(x), \;\; \forall x \in \mathcal{X}. \quad (6)$$

Definition 8. *Let $T : [0,1]^2 \to [0,1]$ be a t-norm. The* ***intersection*** *between the fuzzy sets A and B, both defined with respect to X, results in a fuzzy set $A \cap B = \{(x, f_{A\cap B}(x)) : x \in \mathcal{X}\}$, whose membership function $f_{A\cap B}(x) : \mathcal{X} \to [0,1]$ is given by:*

$$f_{A\cap B}(x) = T(f_A(x), f_B(x)), \forall x \in \mathcal{X}. \quad (7)$$

A characterization of the membership function related to an intersection $A \cap B$ is obtained by applying the algebraic product to the fuzzy sets A and B, given by Eq. (3):

$$f_{A\cap B}(x) = f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \quad (8)$$

Definition 9. *let $S : [0,1]^2 \to [0,1]$ be a t-conorm. An* ***union operation*** *between fuzzy sets A and B, both defined with respect to X, results in a fuzzy set $A \cup B = \{(x, f_{A\cup B}(x)) : x \in \mathcal{X}\}$, whose membership function $f_{A\cup B}(x) : \mathcal{X} \to [0,1]$ is given by:*

$$f_{A\cup B}(x) = S(f_A(x), f_B(x)), \forall x \in \mathcal{X}. \quad (9)$$

A characterization of the fuzzy union can be obtained by the algebraic product in Eq. (3):

$$f_{A\cup B}(x) = f_A(x) + f_B(x) - f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \quad (10)$$

Extending the classical equivalence $\neg(p \to q) \Leftrightarrow p \wedge \neg q$, we obtain the difference operator considering a strong fuzzy negation and an S-implication.

Definition 10. *Let S be t-conorm, N be a strong fuzzy negation and I be an S-implication, A and B be fuzzy sets related to $\mathcal{X}$. The binary operation* ***difference between the fuzzy sets*** *A* ***and*** *B, both defined with respect to X, results in a fuzzy set $A \cup B = \{(x, f_{A-B}(x)) : x \in \mathcal{X}\}$, whose membership function $f_{A-B} : \mathcal{X} \to [0,1]$ is given by:*

$$f_{A-B}(x) = N(S(f_A(x), f_B(x))) \forall x \in \mathcal{X}. \quad (11)$$

And, an example of the difference $A - B$ is given by the standard fuzzy negation of the Reichenbach's implication, expressed as in Eq. (5):

$$f_{A-B}(x) = N_S(S_P(N_S(f_A(x)), f_B(x)) = f_A(x) - f_A(x) \cdot f_B(x), \forall x \in \mathcal{X}. \quad (12)$$

3 Foundations on Quantum Computing

QC considers the development of quantum computers, exploring the phenomena predicted by the *QM* (superposition of states, quantum parallelism, interference, entanglement) for better performance when they are compared to the analogous classical approach [21]. These quantum algorithms are modeled considering some mathematical foundations which describe the phenomenon of *QM*.

3.1 Quantum States

In *QC*, the *qubit* is the basic unit of information, being the simplest quantum system, defined by a state vector, unitary and bi-dimensional, generally described, in the notation of Dirac [21], by the expression

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle. \tag{13}$$

The coefficients α and β are complex numbers corresponding to the amplitudes of the respective states of the computational basis of one-dimensional quantum state space, verifying the normalization condition $|\alpha|^2 + |\beta|^2 = 1$ and ensuring the unitary of the state vector of the quantum system, represented by $(\alpha, \beta)^t$.

The state space of a multiple-dimensional quantum system is obtained by the tensor product of state spaces of corresponding component systems. So, a bi-dimensional quantum system generated by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\varphi\rangle = \gamma|0\rangle + \delta|1\rangle$ is given by the tensor product:

$$|\psi\rangle \otimes |\varphi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle. \tag{14}$$

3.2 Unitary and Controlled Quantum Transformations

The transition of state in a quantum system performed by quantum unitary transformations are associated with ortonormalized matrices of order 2^N, and N being the amount of *qubits* transformation. For instance, the *Pauly X* transformation and $|\psi\rangle$ in Eq.(13) define a vector $X|\psi\rangle$ interpreting a new quantum state:

$$X|\psi\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \tag{15}$$

Additionally, the product tensor of two *Pauly X* transformations is described in Eq (16):

$$X^{\otimes 2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{16}$$

Analogous to transformations of multiple *qubits* which were obtained by the tensor product performed over unitary transformations, the controlled transformations also modify the state of one or more *qubits* considering the current state. The *Toffoli* transformation is a controlled operation performed over 3 *qubits*, which is obtained by a quantum transformations that execute *NOT* (*Pauly X*) to $|\sigma\rangle$ when the current states of first two *qubits* $|\psi\rangle$ and $|\varphi\rangle$ are both assigned as $|1\rangle$. Eq. (17), in the following, presents the matrix structure defining such transformation, when $|\chi\rangle = |\psi\rangle \otimes |\varphi\rangle \otimes |\sigma\rangle$ is the initial state.

$$T|\chi\rangle = \begin{pmatrix} 1\,0\,0\,0\,0\,0\,0\,0 \\ 0\,1\,0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,1\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,1\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,1 \\ 0\,0\,0\,0\,0\,0\,1\,0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \theta \\ \upsilon \\ \sigma \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \\ \theta \\ \sigma \\ \upsilon \end{pmatrix} \quad (17)$$

3.3 Measurement Operations

The reading of the current state of a quantum system is performed by a measurement operator, which is defined based on a set of linear operators M_m, also called projections, acting on quantum state spaces. The index M refers to the possible measurement results. If the state of a quantum system is $|\psi\rangle$ immediately before the measurement, the probability of an outcome occurs is given by [21]:

$$p(|\psi\rangle) = \frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \quad (18)$$

The measurement operators satisfy the completeness relation $\sum_m M_m^\dagger M_m = I$. For one-dimensional quantum systems, the Hermitian (and thus, normal) matrix representation of these operators are given by expressions:

$$M_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1\;0) = \begin{pmatrix} 1\;0 \\ 0\;0 \end{pmatrix} = M_0^\dagger; \quad M_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0\;1) = \begin{pmatrix} 0\;0 \\ 0\;1 \end{pmatrix} = M_1^\dagger.$$

Measurement operators are obviously non-reversible, self-adjoint operators satisfying the completeness relation: $M_0^2 = M_0^2$, $M_1^2 = M_1^2$ and $M_0^\dagger M_0 + M_1^\dagger M_1 = I_2 = M_0 + M_1$.

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ such that $\alpha, \beta \neq 0$. So, the probability of observing $|0\rangle$ and $|1\rangle$ are, respectively, given by:

$$p(|0\rangle) = \langle\phi|M_0^\dagger M_0|\phi\rangle = \langle\phi|M_0|\phi\rangle = |\alpha|^2;$$
$$p(|1\rangle) = \langle\phi|M_1^\dagger M_1|\phi\rangle = \langle\phi|M_1|\phi\rangle = |\beta|^2.$$

Therefore, after the measure the quantum state $|\psi\rangle$ has $|\alpha|^2$ as the probability to be in the classical state $|0\rangle$; and $|\beta|^2$ as the probability to be in the other one, the state $|1\rangle$.

3.4 Bloch's Sphere

In the geometric interpretation of the one-dimensional quantum state $|\psi\rangle$, consider γ and ϕ such that $0 \leq \gamma, \phi < 2\pi$, in order to define α and β by the following expressions $arg(\alpha) = \gamma$ and $\arg(\beta) = \gamma + \phi$, respectively. Thus, we obtain the following amplitudes:

$$\alpha = |\alpha|e^{i\arg(\alpha)} = |\alpha|(\cos\arg(\alpha) + i\sin\arg(\alpha)) = |\alpha|(\cos\gamma + i\sin\gamma)$$
$$\beta = |\beta|e^{i\arg(\beta)} = |\beta|(\cos\arg(\beta) + i\sin\arg(\beta)) = |\beta|(\cos(\gamma + \phi) + i\sin(\gamma + \phi))$$

Applying the coefficient expressions of α and β in $|\psi\rangle$, we obtain that:

$$|\psi\rangle = |\alpha|(\cos\gamma + i\sin\gamma)|0\rangle + |\beta|(\cos(\gamma+\phi) + i\sin(\gamma+\phi))|1\rangle$$
$$= |\alpha|e^{i\gamma}|0\rangle + |\beta|e^{i(\gamma+\phi)}|1\rangle = |\alpha|e^{i\gamma}|0\rangle + |\beta|e^{i\gamma}e^{i\phi}|1\rangle = e^{i\gamma}(|\alpha||0\rangle + |\beta|e^{i\phi}|1\rangle$$

Considering the expressions $\alpha \equiv \cos\frac{\theta}{2}$ and $\beta \equiv e^{i\phi}\sin\frac{\theta}{2}$, for $0 \leq \theta < \pi$, we have the Eq. (19):

$$|\psi\rangle = e^{i\gamma}cos(\frac{\theta}{2})|0\rangle + e^{i\phi}sin(\frac{\theta}{2})|1\rangle, \forall\theta, \gamma, \phi \in \mathbb{R}. \quad (19)$$

However, the term $e^{i\gamma}$, called *global phase*, has no observable physical effect ($e^{i\gamma} = 1$), and by such reason it is not considered in the expression of $|\psi\rangle$, in Eq. (19). Thus, we obtain that:

$$|\psi\rangle = cos(\frac{\theta}{2})|0\rangle + e^{i\phi}sin(\frac{\theta}{2})|1\rangle \quad (20)$$

So, $|\psi\rangle$ can be expressed by polar coordinates, which means, it is parametrized by θ, γ e ϕ, whenever $0 \leq arg(z) < 2\pi$, with $arg(z)$ indicating the main part of the complex argument z.

Proposition 1. *[3, Prop. 3.1] The vector space of elements are expressed by Eq.(20), such that* $0 \leq \theta \leq \frac{\pi}{2}$ *and* $0 \leq \phi < 2\pi$*, are elements of the tri-dimensional vectorial subspace* $\mathcal{C}^2(\mathbb{R})$.

The numbers θ and ϕ identify a point in the tri-dimensional Bloch's sphere, see Fig. 1.

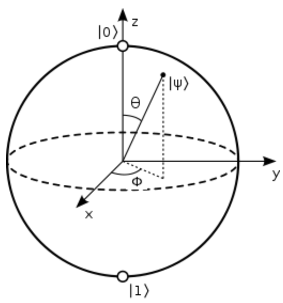


Fig. 1. Bloch's sphere.

In such case, we have that: (i) θ rotates around the Z axis, related to the component basis $|0\rangle$ and $|1\rangle$ defining the one-dimensional quantum state; (ii) ϕ is the angle that the projection of the vector $\overrightarrow{OP}$ on the plane XY makes with the axis X, on the relative phase of the *qubit* $|\phi\rangle$. Furthermore, the angle θ between the vector $\overrightarrow{OP}$ and the $axis\ x$ is related to the "contributions" of the basis state $|0\rangle$ and $|1\rangle$ to the general state of the qubit $|\psi\rangle$. And, the ϕ angle, which is obtained by the projection of the vector over the plane is XY with X axis, corresponds to the relative phase of the *qubit*. Moreover, the relative phase did not change the relative contributions of the qubit, but its basis states may be improve the interference effects used by quantum algorithms. Thus, a state may have the same proportions of $|0\rangle$ and $|1\rangle$ but have different amplitudes due to different relative phase.

4 Modeling fuzzy sets through quantum computing

The description of fuzzy sets from the quantum computing viewpoint considers a fuzzy set$A$ given by $f_A(x)$ membership function, as state in Eq. (2).

Without loss of generality, let $\mathcal{X}$ be a finite subset with cardinality N ($|\mathcal{X}| = N$). Thus, the definitions can be extended to infinite sets, considering in this case, a quantum computer with an infinite quantum register [21].

4.1 Classical Fuzzy States - CFS

Definition 11. *[17, Definition 1] Consider* $\mathcal{X} \neq \emptyset, |\mathcal{X}| = N$, $i \in \mathbb{N}_N = \{1, 2, ..., N\}$ *and a function,* $f : \mathcal{X} \to [0, 1]$. *The state of a* N*-dimensional quantum register, given by:*

$$|s_f\rangle = \bigotimes_{1 \leq i \leq N} [\sqrt{1 - f_A(x_i)}|0\rangle + \sqrt{f_A(x_i)}|1\rangle] \tag{21}$$

is called a ***classical fuzzy state*** *of N-qubits (CFS).*

Reducing denotation, [CFS] and $f(i)$ denote the set of all CFS_S and $f_A(x_i)$, respectively.

Example 1. Consider $\mathbb{N}_1$, a CFSof $|s_f\rangle \in \mathcal{C}^2$ can be described as:

1. classical quantum states:
   - $f_0(1) = 0$, $|s_{f_0}\rangle = \sqrt{1}|0\rangle + \sqrt{0}|1\rangle = |0\rangle$; and $f_1(1) = 1$, $|s_{f_1}\rangle = \sqrt{0}|0\rangle + \sqrt{1}|1\rangle = |1\rangle$.
2. superposition quantum states:
   - $f(1) = a \in]0, 1[$ and $|s_f\rangle = \sqrt{a}|1\rangle + \sqrt{1-a}|0\rangle$.
   Thus, when $a = \frac{1}{2}$, it results that: $|s_{f_2}\rangle = \frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)$; and, when $a = \frac{1}{3}$, we obtained that $|s_{f_3}\rangle = \frac{\sqrt{3}}{3}(\sqrt{2}|0\rangle + |1\rangle)$.

Example 2. Consider $\mathbb{N}_2$, a CFSs $|s_f\rangle \in \mathcal{C}^4$ can also be described as:

1. classical quantum states:
   - $f_4(1) = f_4(2) = 1$, $|s_{f_4}\rangle = |1\rangle \otimes |1\rangle = |11\rangle$;
   - $f_5(1) = 1$ and $f_5(2) = 0$, $|s_{f_5}\rangle = |1\rangle \otimes |0\rangle = |10\rangle$;

- $f_6(1) = 0$ and $f_6(2) = 1$, $|s_{f_6}\rangle = |0\rangle \otimes |1\rangle = |01\rangle$;
- $f_7(1) = f_7(2) = 0$, $|s_{f_7}\rangle = |0\rangle \otimes |0\rangle = |00\rangle$.

2. superposition of quantum states:
   - $f(1) = f(2) = a \in]0,1[$, it results that: $|s_f\rangle = (\sqrt{a}|1\rangle + \sqrt{1-a}|0\rangle) \otimes (\sqrt{a}|1\rangle + \sqrt{1-a}|0\rangle)$, which means that, $|s_f\rangle = (1-a)|00\rangle + \sqrt{a-a^2}|01\rangle + \sqrt{a-a^2}|10\rangle + a|11\rangle$. When $a = \frac{1}{2}$, $f_8(1) = f_8(2) = \frac{1}{2}$, it results that $|s_{f_8}\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.
     If $a = \frac{1}{3}$, $f_9(1) = f_9(2) = \frac{1}{3}$ it implies that $|s_{f_9}\rangle = \frac{1}{3}(2|00\rangle + \sqrt{2}|01\rangle + \sqrt{2}|10\rangle + |11\rangle)$.
   - If $f(1) = a$, $f(2) = b$ and $a, b \in]0,1[$, then it results that $|s_f\rangle$ is given by Eq. (22):

$$|s_f\rangle = (\sqrt{a}|1\rangle + \sqrt{1-a}|0\rangle) \otimes (\sqrt{b}|1\rangle + \sqrt{1-b}|0\rangle)$$
$$= \sqrt{(1-a)(1-b)}|00\rangle + \sqrt{b(1-a)}|01\rangle + \sqrt{a(1-b)}|10\rangle + \sqrt{ab}|11\rangle \quad (22)$$

In addition, if $a = \frac{1}{2}$, $b = \frac{1}{3}$, we have that $|s_{f_{10}}\rangle = \frac{\sqrt{6}}{6}(\sqrt{2}|00\rangle + |01\rangle + \sqrt{2}|10\rangle + |11\rangle$.

In the above examples, from a membership function f, each element in the $f[\mathcal{X}]$ image-set defines a quantum register. In other words, a canonical orthonormal basis in $\otimes^N \mathcal{C}$ identifies a classical quantum register of N-*qubit* corresponds to the set of membership functions of crisp subsets. Thus, one can describe the classical state of the register $|1100\ldots|0\rangle$ of N qubits when $f(1) = f(2) = \{1\}$ and $f(X - \{1,2\}) = \{0\}$.

Generalizing, in the following, an state $|s_f\rangle$ in $\mathcal{C}^{2^N}$ is reported:

Definition 12. *[17, Section 3] The CFS of $N-$qubits, $|s_f\rangle \in [CFS]$, can be expanded in $\mathcal{C}^{2^N}$ by Eq. (23):*

$$\begin{aligned}|s_f\rangle = (1-f(1))^{\frac{1}{2}}(1-f(2))^{\frac{1}{2}}\ldots(1-f(n))^{\frac{1}{2}}|00\ldots00\rangle +\\ f(1)^{\frac{1}{2}}(1-f(2))^{\frac{1}{2}}\ldots(1-f(n))^{\frac{1}{2}}|10\ldots00\rangle +\\ f(1)^{\frac{1}{2}}f(2)^{\frac{1}{2}}\ldots(1-f(2))^{\frac{1}{2}}f(n)^{\frac{1}{2}}|11\ldots01\rangle + \cdots +\\ f(1)^{\frac{1}{2}}f(2)^{\frac{1}{2}}\ldots f(n)^{\frac{1}{2}}|11\ldots11\rangle). \quad (23)\end{aligned}$$

Concluding, from the perspective of QC, a fuzzy set consists on a superposition of crisp sets. Each $|s_f\rangle \in [CFS]$ is a representation of a quantum register described as a superposition of crisp sets and generated by the tensor product of non-entangled quantum registers [21].

4.2 Quantum Fuzzy Sets (QFS)

According to [17], it appears that the fuzzy sets are obtained by overlapping quantum states from a conventional fuzzy quantum register. Moreover, from the set of membership functions representing the fuzzy classical states, we obtain a linear combination, formalizing the notion of a fuzzy quantum register. In this context, it may be characterized quantum fuzzy sets:
(i) as quantum superposition of fuzzy subsets, which have different shapes, simultaneously;
(ii) as subsets of entangled superpositions of crisp subsets (or classical fuzzy sets).

Proposition 2. *[17, Theorem 1] Consider $N = |\mathcal{X}|$, A as a fuzzy subset. A quantum fuzzy subset related to a fuzzy set A is a point in the quantum states space C^{2^N}.*

Proposition 3. *[17, Theorem 2] Let $f, g : X \to [0,1]$ be membership functions with respect to $\mathcal{X}$. The classical fuzzy sets $|s_f\rangle$ and $|s_g\rangle$ are mutually orthonormal CFSs if and only if there exists $x \in \mathcal{X}$ such that either $f(x) = 0$ and $g(x) = 1$ or the converse, $f(x) = 1$ and $g(x) = 0$.*

By Proposition 3, a pair of $|s_f\rangle$ and $|s_g\rangle$ in $[CFS]$ are mutual orthogonal CFSs if and only if there exists $x \in X$ such that $f(x) \cdot g(x) = 0$. In Eq (23), a quantum state $|s_f\rangle$ in $\mathcal{C}^{2^N}$ is characterized, when all vectors are two by two orthonormal elements of a base in $\mathcal{C}^{2^N}$). For further specifications, see [21], [14] and [11].

Definition 13. *Consider $f_i : X \to [0,1]$, $i \in \{1, ..., k\}$, as a collection of membership function generating fuzzy subsets A_i and $\{|s_{f_1}\rangle, \ldots, |s_{f_k}\rangle\} \subseteq [CFS]$, such that their components are two by two orthonormal vectors. Let $\{c_1, \ldots, c_k\} \subseteq \mathcal{C}$. A **quantum fuzzy set** (QFS) $|s\rangle$is defined by the linear combination:*

$$|s\rangle = c_1|s_{f_1}\rangle + \ldots + c_k|s_{f_k}\rangle. \tag{24}$$

[CFQ] denotes the set of all $CFQs$. From Def. 13, a fuzzy quantum state of a N-dimensional quantum register, as described by Eq.(24), can be entangled or not, depending on the family of classical fuzzy states $|s_{f_i}\rangle$ and the set C_i of chosen amplitudes.

It should be emphasized that, in Def. 13, non-entangled fuzzy states can be transformed into classical fuzzy states, by image of rotations on the Bloch's sphere axis (such as rotations of the meridian to achieve a zero phase), see details in [14].

4.3 Entangled Quantum States

The amplitudes of the states of multi-dimensional quantum systems are regulated by the normalization condition, which is not always obtained from the tensor product of corresponding states of qubits (the computational basis states). When this occurs, the quantum system is said to be entangled [16]. In this section we apply the definition of fuzzy quantum state in order to obtain a characterization of entangled states. Consider the classical states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$ presented in Remark 2 as basis vectors of a bi-dimensional quantum system. By Def. 13, an entangled state can be considered as a **quantum fuzzy state**.

As an illustration, in the following, two entangled quantum state are presented:

(*i*) $|s_\alpha\rangle = \alpha_1|00\rangle + \alpha_2|11\rangle$ is a linear combination defined over the classical fuzzy states $\{|00\rangle, |11\rangle\}$, where $\alpha_1, \alpha_2 \in \mathcal{C}$ and $\alpha_1^2 + \alpha_2^2 = 1$; and, analogously,

(*ii*) $|s_\beta\rangle = \beta_1|01\rangle + \beta_2|10\rangle$ is another linear combination defined over the classical fuzzy states $\{|01\rangle, |10\rangle\}$, whenever $\beta_1, \beta_2 \in \mathcal{C}$ and $\beta_1^2 + \beta_2^2 = 1$.

Entangled quantum superposition states can also be represented from Defs. 11 and 13. Note the state $|s_\gamma\rangle = |s_\alpha\rangle \otimes |s_{f_2}\rangle$, immediately presented in the sequence:

$$\begin{aligned} |s_\gamma\rangle &= (\alpha_1|00\rangle + \alpha_2|11\rangle) \otimes (\frac{\sqrt{2}}{2}(|0\rangle + |1\rangle)) \\ &= \frac{\sqrt{2}}{2}(\alpha_1(|000\rangle + |001\rangle) + \frac{\sqrt{2}}{2}\alpha_2(|110\rangle + |111\rangle)). \end{aligned}$$

Thus, $|s_\gamma\rangle$ is a linear combination performed over the classical states

$$\{|000\rangle, |001\rangle, |110\rangle, |111\rangle\},$$

where, by Eq.(24), it follows that $c_1 = c_2 = \frac{\sqrt{2}}{2}\alpha_1$, $c_3 = c_4 = c_5 = c_6 = 0$ e $c_7 = c_8 = \frac{\sqrt{2}}{2}\alpha_2$.

5 Modeling Fuzzy Set Operations from Quantum Transformations

According to [17], fuzzy sets can be obtained by quantum superposition of classical fuzzy states associated with a quantum register. Thus, interpretations relate to the fuzzy operations as complement and intersection are obtained from the NOT and AND quantum transformations. Extending this approach, other operations are introduced, such as union, difference and fuzzy implication, which may derived from interpretations of OR, DIV and IMP quantum operators.

For model, implement and validate these constructions from fuzzy quantum registers we make use of the visual programming environment VPE-qGM. It provides interpretations of the quantum memory, quantum processes and computations related to transition quantum states, which are obtained from the simulation of the corresponding quantum states and quantum transformations. For that, consider the membership functions $f, g : \mathcal{X} \to [0,1]$ obtained according with Eq. (21) and by a pair $(|s_{f_i}\rangle, |s_{g_i}\rangle)$ of CFS, given as:

$$|s_{f_i}\rangle = \sqrt{f(i)}|1\rangle + \sqrt{1-f(i)}|0\rangle, \tag{25}$$

$$|s_{g_i}\rangle = \sqrt{g(i)}|1\rangle + \sqrt{1-g(i)}|0\rangle, \forall x_i \in \mathcal{X}. \tag{26}$$

5.1 Fuzzy Complement Operator

The interpretation of the complement of a fuzzy set, the standard negation is obtained by the NOT operator related to a multi-dimensional quantum systems. The action of the NOT operator is given by the expression:

$$\begin{aligned} NOT(|s_{f_i}\rangle) &= NOT(\sqrt{f(i)}|1\rangle + \sqrt{(1-f(i))}|0\rangle) \\ &= \sqrt{1-f(i)}|1\rangle + \sqrt{f(i)}|0\rangle \end{aligned} \tag{27}$$

The complement operator can be applied to the state $|s_f\rangle = \otimes_{1\le i\le N}|s_{f_i}\rangle$,resulting in an N-dimensional quantum superposition of 1-*qubit* states, described as $\mathcal{C}^{2^N}$ in the computational basis, according with the following expression:

$$\begin{aligned} NOT^N(|s_f\rangle) &= NOT(\otimes_{1\le i\le N}(f(i)^{\frac{1}{2}}|1\rangle(1-f(i))^{\frac{1}{2}}|0\rangle)) \\ &= \otimes_{1\le i\le N}((1-f(i))^{\frac{1}{2}}|1\rangle + f(i)^{\frac{1}{2}}|0\rangle) \end{aligned} \tag{28}$$

Now, Eqs. (29) and (30) describes other applications related to the NOT transformation restricted to 2 e 3-*qubits* quantum systems, respectively:

$$NOT_2(|s_{f_1}\rangle|s_{f_2}\rangle) = |s_{f_1}\rangle \otimes NOT|s_{f_2}\rangle \tag{29}$$

$$NOT_{2,3}(|s_{f_1}\rangle|s_{f_2}\rangle|s_{f_3}\rangle) = |s_{f_1}\rangle \otimes NOT|s_{f_2}\rangle \otimes NOT|s_{f_3}\rangle. \tag{30}$$

Such applications will be used, in next sections, in order to model other operations as fuzzy implications and fuzzy difference.

5.2 Fuzzy Intersection Operator

Consider the pair $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ of quantum states given by Eqs. (25) and (26), respectively. Let Eq. (17) be the matrix expression of a Toffoli gate T, an 3-*qubits* quantum transformation and Eqs. (25) and (26) be a description of membership functions $f, g : \mathcal{X} \rightarrow [0,1]$ related to an element $x_i \in \mathcal{X}$. Fuzzy intersection operator is modelled by an ***AND* operator** given by:

$$\begin{aligned} AND(|s_{f_i}\rangle, |s_{g_i}\rangle) &= T(|s_{f_i}\rangle, |s_{g_i}\rangle, |0\rangle) \\ &= T\left(\sqrt{f(i)}|1\rangle + \sqrt{1-f(i)}|0\rangle), \sqrt{g(i)}|1\rangle + \sqrt{1-g(i)}|0\rangle, |0\rangle\right) \\ &= \left(\sqrt{f(i)}|1\rangle + \sqrt{1-f(i)}|0\rangle\right) \otimes \left(\sqrt{g(i)}|1\rangle + \sqrt{1-g(i)}|0\rangle\right) \\ &\quad \otimes \left(\sqrt{f(i)g(i)}|1\rangle + \sqrt{(1-f(i))g(i)}|0\rangle\right) \end{aligned} \quad (31)$$

By the tensor product distributivity related to sum in Eq. (31), the next expression holds:

$$\begin{aligned} AND(|s_{f_i}\rangle, |s_{g_i}\rangle) = \sqrt{f(i)g(i)}|111\rangle| + \sqrt{f(i)(1-g(i))}|100\rangle + \\ (\sqrt{(1-f(i))g(i)}|010\rangle + \sqrt{(1-f(i))(1-g(i))}|000\rangle. \end{aligned} \quad (32)$$

Thus, a measurement performed over the third *qubit* ($|1\rangle$) in the quantum state expressed by Eq. (32), provides an output $|S_1'\rangle = |111\rangle$, with probability $p = f(i) \cdot g(i)$. Then, for all $i \in X$, $f(i)$ and $g(i)$ indicate the probability of $x_i \in \mathcal{X}$ is in the fuzzy set defined by $f_A(x) : \mathcal{X} \rightarrow U$ and $g_A(x) : \mathcal{X} \rightarrow U$, respectively. And then, $f(i) \cdot g(i)$ indicates the probability of x_i is in the intersection of such fuzzy sets. Analogously, a measurement of third *qubit* ($|0\rangle$) in the quantum state given by Eq. (32), returns an output state given as:

$$\begin{aligned} |S_2'\rangle = \frac{1}{\sqrt{(1-f(i))g(i)}}(\sqrt{f(i)(1-g(i))}|100\rangle + \\ \sqrt{1-f(i)}(\sqrt{g(i)}|010\rangle + \sqrt{1-g(i)}|000\rangle) \end{aligned} \quad (33)$$

with probability $p = 1 - f(i) \cdot g(i)$. In this case, an expression of the complement of the intersection between fuzzy sets A and B is given by $1 - p = f(i) \cdot g(i)$. This probability indicates the non-membership degree of x is in the fuzzy set $A \cap B$. We also conclude that, by Eq. (32), it corresponds to the standard negation of product t-norm, e.i., the standard negation of algebraic product [13].

Consider the quantum state $|s_{f_2}\rangle \otimes |s_{f_3}\rangle \otimes |0\rangle$, according with Eq. (34):

$$|S\rangle = \frac{\sqrt{12}}{6}|000\rangle + \frac{\sqrt{6}}{6}|010\rangle + \frac{\sqrt{12}}{6}|100\rangle + \frac{\sqrt{6}}{6}|110\rangle \quad (34)$$

A simulation is modelled and performed in the *VPE-qGM* environment according with the specification of the intersection operation of fuzzy sets described in Eq. (31) and considering the quantum state $|S\rangle$ in Eq. (34). It is illustrated in Fig. 2(a). In this case, after a measurement, two possible situations are held:

- $|S'_1\rangle = |111\rangle$, with probability $p = 17\%$;
- $|S'_2\rangle = \frac{\sqrt{72}}{6\sqrt{5}}|000\rangle + \frac{\sqrt{36}}{6\sqrt{5}}|010\rangle + \frac{\sqrt{72}}{6\sqrt{5}}|100\rangle$, with probability $p = 83\%$.

See in Fig. 2(a), the quantum state $|S'_2\rangle$ randomly generated in the VPE-qGM.

5.3 Fuzzy Union Operator

Let $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ be quantum states. The union of fuzzy sets is modelled by the ***OR* operator**, based on the expression:

$$\begin{aligned} OR|s_{f_i}\rangle, |s_{g_i}\rangle) &= NOT^3(AND(NOT|s_{f_i}\rangle, NOT|s_{g_i}\rangle)) \\ &= NOT^3(T(NOT|s_{f_i}\rangle, NOT|s_{g_i}\rangle, |0\rangle)) \end{aligned}$$

Then, when $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ are given by Eqs. (25) and (26), respectively, Eq. (35) holds:

$$\begin{aligned} OR|s_{f_i}\rangle, |s_{g_i}\rangle) = NOT^3(T(&\sqrt{f(i)g(i)}|000\rangle + \sqrt{f(i)(1-g(i))}|010\rangle + \\ &\sqrt{(1-f(i))g(i)}|100\rangle + \sqrt{(1-f(i))(1-g(i))}|110\rangle)). \end{aligned} \quad (35)$$

Now, applying the Toffoli and negation transformations we have that:

$$\begin{aligned} OR(|s_{f_i}\rangle, |s_{g_i}\rangle) = &\sqrt{1-f(i)}(\sqrt{1-g(i)}|000\rangle + \\ &\sqrt{g(i)}|011\rangle) + \sqrt{f(i)}(\sqrt{1-g(i)}|101\rangle + \sqrt{g(i)}|111\rangle). \end{aligned} \quad (36)$$

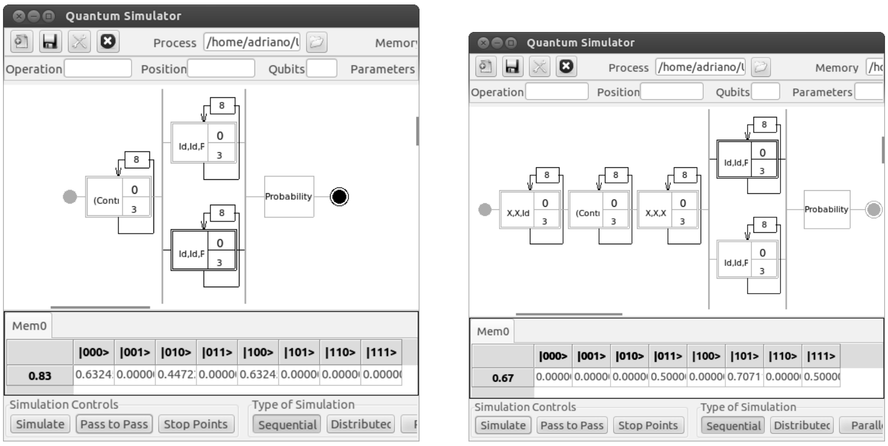
Observe that, a measure performed on third *qubit* of the last expression of the quantum state $OR(|s_{f_i}\rangle, |s_{g_i}\rangle)$ results in the final state:

$$\begin{aligned} |S'_1\rangle = &\frac{1}{\sqrt{g(i)(1-f(i)) + f(i))}}(\sqrt{(1-f(i))g(i)}|011\rangle + \\ &\sqrt{f(i)(1-g(i))}|101\rangle + \sqrt{f(i)g(i)}|111\rangle), \end{aligned}$$

with corresponding probability $p = f(i) + g(i) - f(i) \cdot g(i)$ of $x_i \in \mathcal{X}$ is in both fuzzy sets A e B. That expression is related to the algebraic product, reported in Eq.(3)a, see [13]. Additionally, a measure also performed in the third *qubit* (but related to state $|0\rangle$) returns $|S'_2\rangle = |000\rangle$ with probability $\bar{p} = (1 - f(i)) \cdot (1 - g(i)) = 1 - p$, indicating that $x_i \in \mathcal{X}$ is not in such fuzzy sets (neither A nor B).

The modeling, implementation and simulation on *VPE-qGM* were performed according with the description of union operation in Eq. (35) and considering the initial state as defined by Eq. (34). Fig. 2(b) presents, in the interface of VPE-qGM simulator, the final state. Also observe that, after the measurement process, one of two states is able to be reached:

- $|S'_1\rangle = \frac{1}{2}|011\rangle + \frac{\sqrt{2}}{2}|101\rangle + \frac{1}{2}|111\rangle$, with probability $p = 67\%$ (see it in Fig. 2(b));
- $|S'_2\rangle = |000\rangle$, with probability $p = 33\%$.



(a) Intersection of FSs (b) Union of FSs

Fig. 2. Modeling and simulation of fuzzy operations in the VPE-qGM

5.4 Fuzzy Implication Operator

Fuzzy implications, as many other fuzzy connectives, can be obtained by a composition of quantum operations applied to quantum registers. In the following, this paper introduces the expression of the quantum operator denoted by *IMP*, over which an interpretation of Reichenbach implication is obtained.

For that, consider again the pair $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$ of quantum states given by Eqs. (25) and (26), respectively. The ***IMP* operator** is defined by:

$$\begin{aligned} IMP(|s_{f_i}\rangle, |s_{g_i}\rangle) &= NOT_2(AND(|s_{f_i}\rangle, NOT|s_{g_i}\rangle)) \\ &= NOT_2\,(T(|s_{f_i}\rangle, NOT|s_{g_i}\rangle, |0\rangle)) \\ &= NOT_2(T(\sqrt{1-f(i)}(\sqrt{g(i)}|000\rangle + \sqrt{1-g(i)}|010\rangle) + \\ &\quad + \sqrt{f(i)}(\sqrt{(g(i))}|100\rangle + \sqrt{(1-g(i)}|110\rangle))). \end{aligned} \quad (37)$$

And, applying the *Toffoli* and negation quantum transformations, we have that:

$$\begin{aligned} IMP(|s_{f_i}\rangle, |s_{g_i}\rangle) = \sqrt{f(i)(1-g(i))}|100\rangle) + \sqrt{(1-f(i))g(i)}|011\rangle + \\ + \sqrt{(1-f(i))(1-g(i))}|001\rangle + \sqrt{f(i)g(i)}|111\rangle. \end{aligned} \quad (38)$$

Furthermore, by a measure performed over the third *qubit* ($|1\rangle$) in the state defined by Eq. (38) we can get the quantum state:

$$\begin{aligned} |S'_1\rangle = \frac{1}{\sqrt{1-f(i)+f(i)g(i)}}(\sqrt{(1-f(i))(1-g(i))}|001\rangle + \\ + \sqrt{(1-f(i))g(i)}|011\rangle + \sqrt{f(i)g(i)}|111\rangle), \end{aligned}$$

with probability $p = 1 - f(i) + f(i) \cdot g(i)$ that $x_i \in \mathcal{X}$ is in the fuzzy set determined by the fuzzy implication I_{RB}, whose arguments are the fuzzy membership degrees of x_i

related to fuzzy set A and B, respectively. According with Eq. (38), such interpretation is connected to the Reichenbach implication [1].

And, a measure performed on the third *qubit* (now, in the classical state $|0\rangle$) returns $|S_2'\rangle = |100\rangle$ with probability $p = f(i)(1 - g(i))$ that $x_i \in \mathcal{X}$ is not in the fuzzy set determined by I_{RB} fuzzy implication.

Consider $|s_{f_2}\rangle \otimes |s_{f_3}\rangle \otimes |1\rangle$, according with Eq. (34). The modelling, implementation and simulation in the *VPE-qGM* based on the operator described on Eq. (37), is presented in Fig. 3(a). The related possible final results are in the following:

- $|P_1'\rangle = \frac{\sqrt{2}}{2}|001\rangle + \frac{1}{2}|011\rangle + \frac{1}{2}|111\rangle$, with probability $p = 67\%$ (as the case randomly simulated in Fig. 3(a));
- $|P_2'\rangle = |100\rangle$, with probability $p = 33\%$.

5.5 Fuzzy Difference Operator

In this section, we introduce the quantum operator denoted by DIF, in order to provide interpretation to the difference between fuzzy sets based on quantum computing. The DIF operator is modeled by a composition of NOT and IMP quantum transformations, previously presented in Sections 5.1 and 5.4, considering the same initial conditions: quantum states $|s_{f_i}\rangle$ and $|s_{g_i}\rangle$, given by Eqs. (25) e (26), respectively.

The ***DIF* quantum operator** is defined as follow:

$$\begin{aligned} DIF(|s_{f_i}\rangle, |s_{g_i}\rangle) &= NOT_{2,3}(AND(|s_{f_i}\rangle, NOT|s_{g_i}\rangle)) \\ &= NOT_{2,3}(T(|s_{f_i}\rangle, NOT|s_{g_i}\rangle, |1\rangle) \\ &= NOT_{2,3}(T(\sqrt{1-f(i))g(i)}|000\rangle + \sqrt{1-f(i))(1-g(i))}|010\rangle + \\ &\qquad \sqrt{f(i)(g(i))}|100\rangle + \sqrt{f(i)(1-g(i))}|110\rangle)))). \end{aligned} \tag{39}$$

Then, by Eq. (30), the DIF operator can be expressed as:

$$\begin{aligned} DIF(|\psi\rangle, |\phi\rangle) = &\sqrt{(1-f(i))g(i)}|010\rangle + \sqrt{(1-f(i))(1-g(i))}|000\rangle + \\ &+ \sqrt{f(i)(g(i))}|110\rangle + \sqrt{f(i)(1-g(i))}|101\rangle. \end{aligned} \tag{40}$$

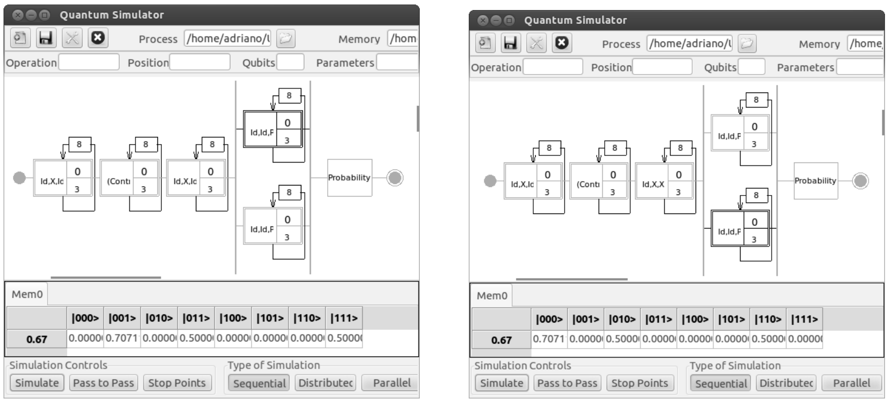
Thus, also in this last case study, we are able to provide an interpretation. After a measure performed over the third *qubit* (in the classical state $|1\rangle$) of the quantum state given by Eq. (40), we can get the final quantum state: $|S_1'\rangle = |101\rangle$, with a probability $p = f(i) - f(i) \cdot g(i)$ that $x_i \in \mathcal{X}$ is the fuzzy set obtained by application of the difference of fuzzy sets A e B, based on both membership degrees of x_i in these corresponding fuzzy sets. And, by other hand, for a measure on the third *qubit* (in state $|0\rangle$) we obtain the state:

$$\begin{aligned} |S_2'\rangle = \frac{1}{\sqrt{(1 - f(i)) + f(i)g(i))}} (&\sqrt{(1 - f(i)(1 - g(i))}|00\rangle + \\ &\sqrt{(1 - f(i))g(i)}|01\rangle + \sqrt{f(i)g(i)}|11\rangle), \end{aligned}$$

with probability $p = 1 - f(i) + f(i)g(i)$ that $x_i \in \mathcal{X}$ is not in the difference of fuzzy sets A e B.

Preserving the configuration of previous case studies, the initial quantum state over that the difference operator is implemented and simulated in *VPE-qGM* is given by the tensor product $|s_{f_2}\rangle \otimes |s_{f_3}\rangle \otimes |1\rangle$, according to Eq. (34).

See Fig. 3(b), giving an illustration of an simulation on VPE-qGM of the DIF operator.



(a) Implication of FSs (b) Difference of FSs

Fig. 3. Modeling and simulation of fuzzy operators ing the VPE-qGM

The both possible results of this simulation are the quantum states in the following:

- $|S_1'\rangle = |101\rangle$, with probability $p = 33\%$;
- $|S_2'\rangle = \frac{\sqrt{2}}{2}|000\rangle + \frac{1}{2}|010\rangle + \frac{1}{2}|110\rangle$, with probability $p = 67\%$ (which is presented in Fig. 3(b) and it was obtained by a randomly simulation on VPE-qGM).

6 Conclusion and Final Remarks

This paper analyses the operations of fuzzy complement and fuzzy intersection as described in [17] but it also implements and simulates them in the VPE-qGM presenting an extension of such construction to other important fuzzy operations. This extension considers the modelling of the following fuzzy operations obtained from quantum operators: union, difference and implications, focusing on the class of S-implications named Reichenbach implications.

The visual approach of the VPE-qGM environment enables the implementation and validation of the description of such fuzzy operations from quantum computing. The description of these operations is based on compositions of controlled and unitary quantum transformations, and the corresponding interpretation of fuzzy operations is obtained by applying operators of projective measurement.

Further work considers the specification of other fuzzy connectives and fuzzy constructors (e. i. automorphisms and reductions) and the corresponding extension of fuzzy methodology from formal structures provided by quantum computing.

As pointed out by [17], this extension may collaborate with the following areas:

– applications on large databases processing, which could take advantage of the parallelism of quantum computing in the implementation of fuzzy inference systems.
– systems whose components have different characteristics with a high degree of correlation but of unknown nature. The level of correlation between the variables could be controlled by appropriate entangled quantum states generated by unitary operators.
– design new algorithms based on amplitude amplification, such as Grover's algorithm used to non-ordered database searches by amplifying the data amplitudes to fit predetermined training data. In this context, related applications are in fuzzy mathematical morphology as well as in the design of fuzzy filter for image processing.

Finally, such growing synergy between QC and FL (see, e.i., [29,25,5,15,27] and [24]) may also contribute to base quantum algorithms considering the abstractions provided by quantum fuzzy sets and related interpretation of fuzzy logic concepts.

References

1. M. Baczyński and J. Drewniak. Monotonic fuzzy implications. *Fuzzy Systems in Medicine, Physica-Verlag, Heidelberg*, pages 90–111, 2000.
2. H. Bustince, P. Burillo, and F. Soria. Automorphism, negations and implication operators. *Fuzzy Sets and Systems*, 134(2):209–229, 2003.
3. L. M. Carvalho, C. Lavor, and V. S. Motta. Caracterizao matemtica e visualizao da esfera de bloch: Ferramentas para computao quntica. *TEMA: Tendncias em Matemtica Aplicada e Computacional*, 8(3):351–360, 2007.
4. C. H. Chen, C. J. Lin, and Lin C.T. An efficient quantum neurofuzzy classifier based on fuzzy entropy and compensatory operation. *Softing computing - A fusion of foundations, methodologies and applications*, 12(6), 2008.
5. Andreas de Vries. Algebraic hierarchy of logics unifying fuzzy logic and quantum logic, 2007. Comment: 59 pages, 17 figures, lecture notes.
6. D. Dubois and H. Prade. *Fundamentals of Fuzzy Sets*. Kluwer Pubs., Boston, 2000.
7. J. Fodor and M. Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Pub., Dordrecht, 1994.
8. N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden. Quantum cryptography. *Reviews of Modern Physics*, pages 145–195, 2002.
9. L. Grover. A fast quantum mechanical algorithm for database search. *Proceedings of the Twenty-Eighth Annual ACM Symposium on Theory of Computing*, pages 212–219, 1996.
10. F. Herrera. Applicability of the fuzzy operators in the design of fuzzy logic controllers. Technical report, 1995.
11. S. Imre and F. Balazs. *Quantum Computing and Communications - an Engineering Approach*. John Wiley & Sons, NJ, 2005.
12. E. P. Klement and M. Navara. *Fuzzy Sets, Logics and Reasoning About Knowledge*, volume 15 of *Applied logics*, chapter Propositional fuzzy logics based on Frank t-norms: A comparison, pages 17–38. Kluwer Academic Publishers, Dorbrech, 1999.
13. Georg J. Klir and B. Yuan. *Fuzzy Sets and Fuzzy Logic; Theory and Applications*. Prentice Hall, Upper Saddle River, N. Y., 1995.

14. E.H. Knill and M.A. Nielsen. Theory of quantum computation, 2002.
15. V. Kreinovich, L. Kohout, and E. Kim. Square root of not: A major difference between fuzzy and quantum logics, 2009.
16. A. F. Lima and B. L. Jnior. *Computao Quntica, noes bsicas utilizando a linguagem de circuitos qunticos*. EDUFCG, PR, 2007.
17. M. A. Mannucci. Quantum fuzzy sets: Blending fuzzy set theory and quantum computation. *CoRR*, abs/cs/0604064, 2006.
18. A. Maron, A. Pinheiro, R. Reiser, and M. Pilla. Consolidando uma infraestrutura para simulao quntica distribuda. In *Anais da ERAD 2011*, pages 213–216. SBC/Instituto de Informtica UFRGS, 2011.
19. A. Maron, A. Pinheiro, R. Reiser, A. Yamin, and M. Pilla. Ambiente vpe-qgm: Em direo a uma nova abordagem para simulaes qunticas. *Rev. CCEI*, 14:29–46, 2010.
20. G. Melnichenko. Energy discriminant analysis, quantum logic, and fuzzy sets. *J. Multivariate Analysis*, 101(1), 2010.
21. Michael A. Nielsen and Isaac L. Chuang. *Computao Quntica e Informao Quntica*. Bookman, 2003.
22. S. R. Nikam, P. J. Nikumbh, and S. P. Kulkarni. Fuzzy logic and neuro-fuzzy modeling. *International Journal of Computer Applications*, (4):22–31, 2012.
23. S. Pironio, A. Acn, S. Massar, A. Boyer de la Giroday, D.N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T.A. Manning, and C. Monroe. Random numbers certified by bell's theorem. *Nature*, 464:1021–1024, 2010.
24. A. Raghuvanshi and M. A. Perkowski. Fuzzy quantum circuits to model emotional behaviors of humanoid robots. In *IEEE Congress on Evolutionary Computation*, pages 1–8. IEEE, 2010.
25. G. G. Rigatos. Quantum wavepackets in fuzzy automata and neural associative memories. *International Journal of Modern Physics*, 18(10):1551–1569, 2006.
26. G. G. Rigatos and S. G. Tzafestas. Parallelization of a fuzzy control algorithm using quantum computation. *IEEE-FS*, 10:451–460, 2002.
27. R. Seising. On classical, fuzzy classical, quantum and fuzzy quantum systems. pages 1338–1342, 2009.
28. P. Shor. Polynomial-time algoritms for prime factorization and discrete logarithms on a quantum computer. *SIAM Journal on Computing*, 1997.
29. S. Yun, L. Yongming, and C. Maoyin. *Constructions of Some Quantum Structures and Fuzzy Effect Space*, volume 43. Springer Science, New York, 2004.