Simultaneous Fuzzy Segmentation of Multiple Objects

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Abstract

Fuzzy segmentation is a technique that assigns to each element in an image (corrupted by noise and/or shading) a grade of membership in an object (which is believed to be contained in the image). In an earlier work the first two authors extended this concept by presenting and illustrating an algorithm which simultaneously assigns to each element in an image a grade of membership in each one of a number of objects (which are believed to be contained in the image). In this paper we establish the correctness of this algorithm (in the sense of producing an output that is uniquely specified by a desirable mathematical property) and present a further example of its use in medical imaging. We also compare our method with two recently published related methods.

Key words:
fuzzy segmentation, fuzzy graph, greedy algorithms

1 Introduction

Digital image segmentation is the process of assigning distinct labels to different objects in an image. The level of detail indicated by the labeling is related to the application at hand. To perform object identification in digital or continuous, moving or still images, humans make use of high-level reasoning and knowledge, as well as of different visual cues, such as shadowing, occlusion, parallax motion and the relative sizes of objects. Aside from the difficulty of inserting this type of reasoning into a computer program, the task of segmenting an object from a background in an image becomes particularly hard for a computer when, instead of the brightness values, what distinguishes the object from the background is some textural property, or when the image is corrupted by noise and/or inhomogeneous illumination. One concept that has been successfully used to achieve segmentation in such corrupted images is fuzzy connectedness, as can be seen in [1,2] and their references. Our approach here is a generalization of the one advocated in [3] (based on the work of [4]) to arbitrary digital spaces [5] and simultaneous multiple object segmentation [6]. We present below, for the first time, proofs of the claims in [6] regarding the nature of the output of the algorithm proposed there.

We also discuss the relationship of the methodology proposed by us in [6] (and reproduced here) to the alternative methods to fuzzy segmentation that have appeared in a recent paper by Udupa, Saha, and Lotufo [7].

In our previous work [6] we carefully motivated our approach and gave a precise mathematical description of it. Much of this description is reproduced below in the next section, but prior to getting into the theory we give a picturesque (but nevertheless precise) description of our algorithm. (Since the main aim of our description is understandability and one of the aims of the actual algorithm is computational efficiency, there are some differences between the description and the actual algorithm as stated later; however the two processes produce the same result and in essentially the same manner.)

Our model for describing the algorithm takes the form of a military exercise. It involves a number of castles such that there is a one-way road from every castle to every other castle. There are also a number of armies. Each road from a castle to another one has an *affinity* for each army, this is measured by a nonnegative integer (the lower this integer, the more difficult it is for that army to travel along that road). The affinities of the roads for the various armies are fixed for the duration of the exercise. We also fix an integer MAX that is greater than or equal to any of the affinities of any of the roads for any of the armies.

The purpose of the exercise is to see how the final territories of each of the armies depend on their initial arrangements. Since we are discussing an algorithm here, no initiative is to be taken by the individual armies: they have to follow the rules of combat to be described momentarily.

All through the exercise each castle will have have a *strength* assigned to it, this strength is an integer in the range [0, ..., MAX]. The strength of a castle may change as the exercise proceeds. Also, at any time, each castle may be occupied by one or more of the armies.

The exercise starts by distributing the soldiers of the armies into some of the castles and assigning to those castles which have soldiers in them the strength MAX. We say that this distribution of armies and strengths describes the situation at the start of Iteration 0.

The exercise proceeds in discrete iterative steps. The following gets done during Iteration i. Those soldiers (and only those soldiers) which occupy a castle of strength MAX$-i$ will try to increase the territory of their army. They will send units from their castle toward all the other castles. When these units arrive at another castle, their *power* will be defined as the minimum of MAX$-i$ and the affinity for their army of the road from the originally occupied castle to the new one. If the strength of the new castle is greater than the power of any of the armies arriving at it, its strength and occupancy will not change. If no arriving army has greater power than the strength of the new castle, then the strength of the new castle does not change, but it will get occupied also by those arriving armies whose power matches that strength (but not by any of the others). If some of the arriving armies have greater power than the strength of the castle, then the castle will be taken over by those (and only those) arriving armies that have the greatest power, and the strength of the castle is set to the power of the new occupiers. This describes what happens in one iterative step except for one detail: if an army gets to occupy a new castle because its power is MAX$-i$ (this can only happen if the affinity for this army of the road to this castle is at least MAX$-i$), then that army is allowed to to send out units from this new castle as well. (This cannot lead to an infinite loop, since there are only finitely many castles and so it can only happen finitely many time that an army gets to occupy a *new* castle because its power is MAX$-i$.)

The exercise stops at the end of Iteration MAX-1. So altogether there are MAX iterative steps (since we start with Iteration 0). The output of the algorithm provides, for each castle, the strength of the castle and the armies that occupy it at the end of the exercise.

2 Theory

In our very general approach we deal with an arbitrary finite set V, whose elements are referred to as *spels* (short for *spatial elements*). These spels can represent many different things, such as pixels of an image (as in [1,2,3,4]), dots in the plane (as in [8]) or feature vectors (as in [9]). In the picturesque description above, V is the set of castles. Furthermore, the theory and algorithm introduced in [6], and further discussed here, are independent of the specifics of the application area, and thus can be applied to data clustering [10] in general. A special choice (some papers on fuzzy segmentation restrict their attention only to Vs of this type, see for example [7]) is when the Vs are of the form

$$V = \left\{ c \,\middle|\, -b_j \leq c_j \leq b_j \text{ for some } b \in \mathbb{Z}_+^n \right\}, \tag{1}$$

where $\mathbb{Z}_+^n$ is the set of n-tuples of positive integers. Throughout this paper we illustrate the methods on a particularly simple V, which we denote it by $\overline{V}$, that is defined by (1) with $n = 1$ and $b_1 = 1$ (i.e., $\overline{V} = \{(-1), (0), (1)\}$).

We desire to partition V into a number of objects, but in a fuzzy way; i.e., in addition to each spel being judged to belong to a particular object, it is also assigned a grade of membership in the object (that is, a number between 0 and 1, where 0 indicates that the spel definitely does not belong to the object, and 1 indicates that it definitely does). In the picturesque description above, at the end of the exercise an object consists of the all the castles occupied by one particular army, and the grade of membership of the castle in the object is proportional to its strength. (To make the grade of membership satisfy the requirement that it is not greater than one, we can divide the strength of each castle by MAX.) To formalize such fuzzy partitioning, we introduce the concept of an *M-semisegmentation* (M is the number of objects) of V, which is a function σ that maps each $c \in V$ into an $(M+1)$-dimensional vector $\sigma^c = (\sigma_0^c, \sigma_1^c, \cdots, \sigma_M^c)$, such that $\sigma_0^c \in [0, 1]$ (i.e., it is nonnegative but not greater than 1) and for at least one m in the range $1 \leq m \leq M$ $\sigma_m^c = \sigma_0^c$, and for all other m it is either 0 or σ_0^c. (We point out that this definition of M-semisegmentation allows a spel to belong to more than one object, as long as it has the same grade of membership in all of them. For the exercise described above, σ_0^c is proportional to the strength of the castle c and the mth army occupies that castle if, and only if, $\sigma_m^c = \sigma_0^c$.) We say that σ is an *M-segmentation* if, for every spel c, σ_0^c is positive. An example of a 2-segmentation $\bar{\sigma}$ of $\overline{V}$ is defined by $\bar{\sigma}^{(-1)} = (1, 0, 1)$, $\bar{\sigma}^{(0)} = (1, 1, 0)$ and $\bar{\sigma}^{(1)} = (0.25, 0.25, 0)$; i.e., (-1) is definitely in the second object, (0) is definitely in the first object, and (1) is in the first object with grade of membership 0.25.

The basic concept that we are generalizing here is that of fuzzy connectedness. To every ordered pair (c, d) of spels, we assign a real number not less than 0 and not greater than 1, which we define as the *fuzzy connectedness* of c to d. This provides us with an example of a fuzzy set: the set in question is the set of ordered pairs of spels and the *grade of membership* of (c, d) in this set is the fuzzy connectedness of c to d. In the approach used below, fuzzy connectedness is defined in the following general manner.

We call a sequence of spels a *chain*; its *links* are the ordered pairs of consecutive spels in the sequence. The strength of a link is also a fuzzy concept (i.e., for every ordered pair (c, d) of spels, we assign a real number not less than 0 and not greater than 1, which we define as the *strength of the link* from c to d). The strength of a link is provided by the appropriate value of a *fuzzy spel affinity* function $\psi : V^2 \to [0, 1]$, i.e., a function that assigns a value between 0 and 1 to every ordered pair of spels in V. (As we illustrate later, for the purpose of fuzzy segmentation of images, fuzzy spel affinities can often be automatically defined based on statistical properties of the links within regions identified by the user as belonging to the object of interest.) The *ψ-strength of a chain* is the ψ-strength of its weakest link; the ψ-strength of a chain with only one spel in it is 1 by definition. A set $U(\subseteq V)$ is said to be ψ-*connected* if, for every pair of spels in U, there is a chain in U of positive ψ-strength from the first spel of the pair to the second. For the picturesque description above, (c, d) denotes the one-way road from castle c to castle d, and an affinity of an army for this road has to be divided by MAX in order to match the definition of a fuzzy spel affinity.

In our approach there are no further restrictions on the definition of fuzzy spel affinity; other researchers restrict them to be reflexive (i.e., $\psi(c, c) = 1$ for all $c \in V$) and symmetric (i.e., $\psi(c, d) = \psi(d, c)$ for all $c, d \in V$) [7]. Examples of such reflexive and symmetric fuzzy spel affinities are $\overline{\psi_1}$ and $\overline{\psi_2}$, defined by the additional conditions $\overline{\psi_1}((-1), (0)) = 0.5$, $\overline{\psi_1}((0), (1)) = 0.25$ and $\overline{\psi_1}((-1), (1)) = 0$, and $\overline{\psi_2}((-1), (0)) = \overline{\psi_2}((0), (1)) = 0.5$ and $\overline{\psi_2}((-1), (1)) = 0$. The $\overline{\psi_1}$-strength of the chain $\langle(-1), (0), (1)\rangle$ in $\overline{V}$ is 0.25, its $\overline{\psi_2}$-strength is 0.5, and $\overline{V}$ is both $\overline{\psi_1}$-connected and $\overline{\psi_2}$-connected.

If one wants to segment multiple objects, it is reasonable to define different fuzzy spel affinities for each one of them. (This corresponds to the idea of each army having its own affinity for each one-way road.) In general, an M-*fuzzy graph* is a pair (V, Ψ), where V is a finite set and $\Psi = (\psi_1, \cdots \psi_M)$ with ψ_m (for $1 \leq m \leq M$) being a fuzzy spel affinity such that V is ϕ_Ψ-connected, where $\phi_\Psi(c, d) = \min_{1 \leq m \leq M} \ \psi_m(c, d)$. An example of a 2-fuzzy graph is $\left(\overline{V}, \overline{\Psi}\right)$, where $\overline{\Psi} = \left(\overline{\psi_1}, \overline{\psi_2}\right)$. An M-fuzzy graph can be used to totally specify the aspects of the castles and the roads connecting them that are relevant to the rules of combat given above. The requirement of ϕ_Ψ-connectedness is not needed for this purpose, it is introduced only so that we can prove below that the outcome of the exercise is in fact an M-segmentation (and not just an M-semisegmentation as it would be without this requirement).

For an M-semisegmentation σ of V and for $1 \leq m \leq M$, we define the chain $\langle c^{(0)}, \cdots, c^{(K)}\rangle$ to be a σm-*chain* if $\sigma_m^{c^{(k)}} > 0$, for $0 \leq k \leq K$. Furthermore, for $U \subseteq V$, $W \subseteq V$ and $c \in V$, we use $\mu_{\sigma,m,U,W}(c)$ to denote the maximal ψ_m-strength of a σm-chain in U from a spel in W to c. (This is 0 if there is no such chain.)

Theorem 1 *If $G = (V, \Psi)$ is an M-fuzzy graph and, for $1 \leq m \leq M$, V_m is a subset (of* seed *spels) of V such that at least one of these subsets is nonempty, then*

(i) there exists an M-semisegmentation σ of V with the following property: for every $c \in V$, if for $1 \leq n \leq M$

$$s_n^c = \begin{cases} 1, & \text{if } c \in V_n, \\ \max_{d \in V} \left(\min\left(\mu_{\sigma,n,V,V_n}(d), \psi_n(d, c)\right)\right), & \text{otherwise}, \end{cases} \tag{2}$$

then for $1 \leq m \leq M$

$$\sigma_m^c = \begin{cases} s_m^c, \text{ if } s_m^c \geq s_n^c, \text{ for } 1 \leq n \leq M, \\ 0, \qquad\qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{3}$$

(ii) this M-semisegmentation σ *is unique; and*
(iii) it is an M-segmentation.

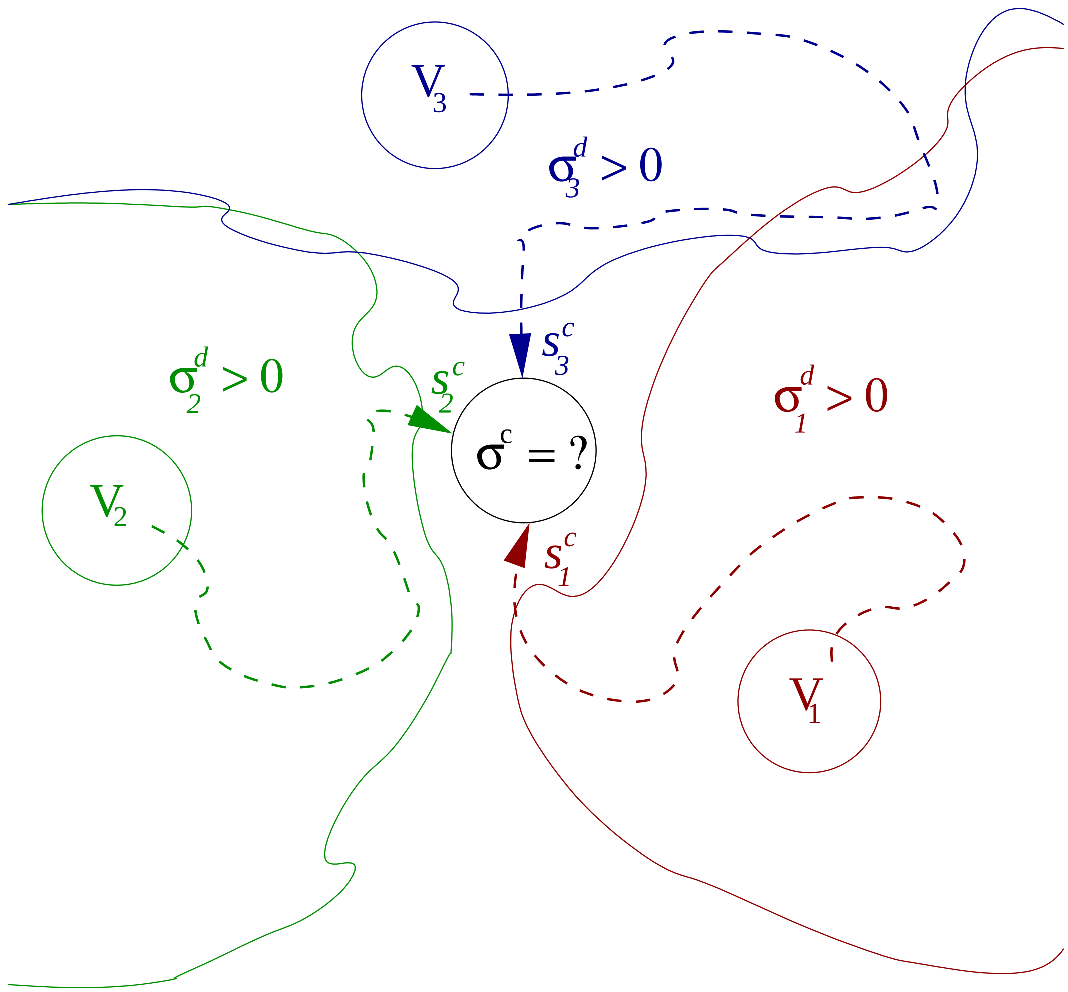

Figure 1. Illustration of the desirability of the M-segmentation whose existence (and uniqueness) is guaranteed by Theorem 1.

Before discussing the validity of Theorem 1, let us discuss in less mathematical terms what it says. The property stated in Theorem 1 is a reasonable one, as we can see in Fig. 1. Suppose, as in Fig. 1, that c is an arbitrary spel and that σ^d is known for all other spels d. Then, for $1 \leq n \leq M$ (M=3 in Fig. 1), the s_n^c of (2) is the maximal ψ_n-strength of a chain $\langle d^{(0)}, \ldots, d^{(L)}, c\rangle$ from a seed spel in V_n to c such that $\sigma_n^{d^{(l)}} > 0$ (i.e., $d^{(l)}$ belongs to the nth object), for $0 \leq l \leq L$. (s_n^c is defined to be 0 if there is no such chain.) Intuitively, the

mth object (the red, green or blue object) can “claim” that c belongs to it if, and only if, s_m^c is maximal. This is indeed how things get sorted out in (3): σ_m^c has a positive value only for such objects. Furthermore, this property tells us how any one spel relates to the various objects, provided that we have such information for the other spels: for a fixed spel c we can work out the values of the s_n^c using (2) and what we request is that, at that spel c, (3) be satisfied. What Theorem 1 says that there is one, and only one, M-semisegmentation which satisfies this reasonable property simultaneously everywhere, and that this M-semisegmentation is in fact an M-segmentation.

We now illustrate Theorem 1 for the above-specified 2-fuzzy graph $\left(\overline{V}, \overline{\Psi}\right)$. If we choose the sets of seed spels to be $\overline{V_1} = \{(0)\}$ and $\overline{V_2} = \{(-1)\}$, then we get exactly the $\overline{\sigma}$ defined above. Suppose, for example, that we have been informed that $\overline{\sigma}^{(-1)} = (1, 0, 1)$ and $\overline{\sigma}^{(0)} = (1, 1, 0)$ and we wish to use Theorem 1 to determine $\overline{\sigma}^{(1)}$. We find that $s_1^{(1)} = 0.25$ (obtained by the choice $d = (0)$) and $s_2^{(1)} = 0$ (if we choose in (2) d to be (-1), then $\overline{\psi_2}((-1), (1)) = 0$, if we choose it to be (0), then $\mu_{\overline{\sigma},2,\overline{V},\overline{V_1}}(0) = 0$ since there is no $\overline{\sigma}$2-chain containing (0), due to the fact that $\overline{\sigma}_2^{(0)} = 0$. Hence (3) tells us that indeed $\overline{\sigma}^{(1)} = (0.25, 0.25, 0)$. There is something subtle that takes place here: there is a chain $\langle(-1), (0), (1)\rangle$ of $\overline{\psi_2}$-strength 0.5 from the only seed spel of Object 2 to (1), while the maximal $\overline{\psi_1}$-strength of any chain from the only seed spel of Object 1 to (1) is only 0.25; nevertheless, (1) is assigned to Object 1 by Theorem 1, since the fact that (0) is a seed spel of Object 1 prevents it (for the given $\overline{\Psi}$) from being also in Object 2, and so the chain $\langle(-1), (0), (1)\rangle$ is “blocked” from being a $\overline{\sigma}$2-chain.

The proof of Theorem 1(i) shown below has not been published before, while the proofs of Theorem 1(ii) and Theorem 1(iii) were originally published in [6].

Proof of Theorem 1(i)

In this existence proof we provide an inductive definition that resembles both the picturesque description of the previous section and the actual algorithm of the next section. The reader should however be warned: this inductive definition is not strictly identical to the algorithm (it was designed to make our proof simple, while the algorithm was designed to be efficient). In the next section we describe the relationship between the inductive definition and the actual algorithm.

Let $R = \{1\} \cup \{\psi_m(c, d) > 0 \,|\, 1 \leq m \leq M,\ c, d \in V\}$. R is a finite set of real numbers from $(0, 1]$, and so its elements can be put into a strictly decreasing order $1 = {}^1r > {}^2r > \cdots > {}^{|R|}r > 0$. We define inductively a sequence of M-semisegmentations ${}^1\sigma, {}^2\sigma, \cdots, {}^{|R|}\sigma$ and a sequence ${}^2U, \cdots, {}^{|R|}U$ of subsets of V as follows.

For any $c \in V$ and $1 \leq m \leq M$,

$$ {}^{1}\sigma_m^c = \begin{cases} 1, \text{ if there is a chain of } \psi_m\text{-strength 1 from a seed in } V_m \text{ to } c, \\ 0, \hfill \text{otherwise.} \end{cases} \tag{4} $$

(Here, and later, the definition of ${}^{i}\sigma_0^c$ implicitly follows from the fact that ${}^{i}\sigma$ is an M-semisegmentation.)

For $1 \leq i \leq |R|$, we define

$$ {}^{i}U = \left\{ c \,|\, {}^{i}\sigma_0^c \geq {}^{i}r \right\}. \tag{5} $$

For $1 < i \leq |R|$, $c \in V$ and $1 \leq m \leq M$, we define

$$ {}^{i}\sigma_m^c = \begin{cases} {}^{(i-1)}\sigma_m^c, & \text{if } c \in {}^{(i-1)}U, \\ {}^{i}r, & \text{if there is a chain } \left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle \text{ of } \psi_m\text{-strength} \\ & {}^{i}r \text{ such that } c^{(0)} \in {}^{(i-1)}U,\ {}^{(i-1)}\sigma_m^{c^{(0)}} > 0, \\ & c^{(K)} = c \text{ and, for } 1 \leq k \leq K,\ c^{(k)} \notin {}^{(i-1)}U, \\ 0, & \text{otherwise.} \end{cases} \tag{6} $$

It is obvious from these definitions that ${}^{i}\sigma$ is an M-semisegmentation, for $1 \leq i \leq |R|$. We now demonstrate the definitions on the already repeatedly discussed 2-fuzzy graph $\left(\overline{V}, \overline{\Psi}\right)$ with seed sets $\overline{V_1} = \{(0)\}$ and $\overline{V_2} = \{(-1)\}$. For this case $R = \{1,\ 0.5,\ 0.25\}$. It immediately follows from (4) that ${}^{1}\sigma^{(-1)} = (1,\ 0,\ 1)$, ${}^{1}\sigma^{(0)} = (1,\ 1,\ 0)$, and ${}^{1}\sigma^{(1)} = (0,\ 0,\ 0)$. It turns out that ${}^{2}\sigma = {}^{1}\sigma$. This is because ${}^{1}U = \{(-1),\ (0)\}$, and there are no chains starting at either of these spels which satisfy all the conditions listed in the second line of (6). On the other hand, the chain $\langle (0),\ (1) \rangle$ can be used to generate ${}^{3}\sigma$, which is in fact the 2-segmentation specified by the condition of Theorem 1. This is not an accident, we are now going to prove that in general the ${}^{|R|}\sigma$ defined by (4) and (6) satisfies the property stated in Theorem 1(i).

It clearly follows from the definitions (4) and (6) that, for $c \in V$ and $1 \leq m \leq M$, ${}^{|R|}\sigma_m^c \in R \cup \{0\}$. Furthermore, it is also not difficult to see, for $1 < i \leq |R|$, that if $c \in {}^{i}U$, then ${}^{i}\sigma_m^c = {}^{|R|}\sigma_m^c$, and that

$$ {}^{i}U = \left\{ c \,|\, {}^{|R|}\sigma_0^c \geq {}^{i}r \right\}. \tag{7} $$

From these follow the following two properties of the M-semisegmentation ${}^{|R|}\sigma$.

A. For $c \in V$ and $1 \leq m \leq M$, ${}^{|R|}\sigma^c_m = 1$ if, and only if, there is a chain of ψ_m-strength 1 from a seed in V_m to c.

B. For $c \in V$, $1 \leq m \leq M$, and $2 \leq i \leq |R|$, ${}^{|R|}\sigma^c_m = {}^ir$ if, and only if, there is a chain $\left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle$ of ψ_m-strength ${}^ir$ such that $c^{(0)} \in {}^{(i-1)}U$, ${}^{|R|}\sigma^{c^{(0)}}_m > 0$, $c^{(K)} = c$ and, for $1 \leq k \leq K$, $c^{(k)} \notin {}^{(i-1)}U$.

Let $c, d \in V$. We say that (c, d) is *consistent* if, for $1 \leq m \leq M$, ${}^{|R|}\sigma^c_m = {}^{|R|}\sigma^c_0$ implies that one of the following is true:

$$ {}^{|R|}\sigma^d_0 > \min\left({}^{|R|}\sigma^c_0,\, \psi_m(c, d)\right); \tag{8} $$

$$ {}^{|R|}\sigma^d_0 = \min\left({}^{|R|}\sigma^c_0,\, \psi_m(c, d)\right) \text{ and } {}^{|R|}\sigma^d_m = {}^{|R|}\sigma^d_0. \tag{9} $$

We now show that, for all $c, d \in V$, (c, d) is consistent.

To do this, we assume that there is a (c, d) and an m such that ${}^{|R|}\sigma^c_m = {}^{|R|}\sigma^c_0$ and yet neither (8) nor (9) holds and show that this leads to a contradiction. A consequence of our assumption is that at least one of the following must be the case:

$$ {}^{|R|}\sigma^d_0 < \min\left({}^{|R|}\sigma^c_0,\, \psi_m(c, d)\right); \tag{10} $$

$$ {}^{|R|}\sigma^d_0 = \min\left({}^{|R|}\sigma^c_0,\, \psi_m(c, d)\right) \text{ and } {}^{|R|}\sigma^d_m \neq^{|R|} \sigma^d_0. \tag{11} $$

We may assume that ${}^{|R|}\sigma^c_0 > 0$, for otherwise one of (8) or (9) clearly holds. Hence${}^{|R|}\sigma^c_m = {}^{|R|}\sigma^c_0 = {}^ir$, for some $1 \leq i \leq |R|$. From (10) and (11) it follows that ${}^{|R|}\sigma^d_0 \leq {}^ir$. It follows then from (7) that if $i \geq 2$, then neither c nor d is in ${}^{(i-1)}U$.

If $i = 1$,then by **A** there is a chain of ψ_m-strength 1 from a seed in V_m to c. If $i \geq 2$, then by **B** there is a chain $\left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle$ of ψ_m-strength ${}^ir$ such that $c^{(0)} \in {}^{(i-1)}U$, ${}^{|R|}\sigma^{c^{(0)}}_m > 0$, $c^{(K)} = c$ and, for $1 \leq k \leq K$, $c^{(k)} \notin {}^{(i-1)}U$. In either case, if $\psi_m(c, d) \geq {}^ir$, we can extend the chains without losing their just stated properties to d, and then **A** or **B** implies that ${}^{|R|}\sigma^d_m = {}^ir$. It follows that (9) holds, a contradiction. So assume that $\psi_m(c, d) = {}^jr$ for some $j > i$. Since (10) or (11) holds, we get that $d \notin {}^{(j-1)}U$. But $c \in {}^{(j-1)}U$, and so, applying **B** to the chain $\langle c, d \rangle$, we get that ${}^{|R|}\sigma^d_m = {}^jr$. This implies that (9) holds. This final contradiction completes our proof that, for all $c, d \in V$, (c, d) is consistent.

Next we show that, for all $c \in V$ and $1 \leq m \leq M$,

$$ {}^{|R|}\sigma^c_m = \mu_{{}^{|R|}\sigma, m, V, V_m}(c). \tag{12} $$

To simplify the notation, we use in this proof s to abbreviate ${}^{|R|}\sigma^c_m$. Recall that $\mu_{{}^{|R|}\sigma, m, V, V_m}(c)$ denotes the maximal ψ_m-strength of an ${}^{|R|}\sigma m$-chain from a seed in V_m to c. Note that we can assume that $s \in R$, for the alternative is that $s = 0$ in which case there can be no ${}^{|R|}\sigma m$-chain that includes c and so that right hand side of (12) is also 0 by definition. Our proof will be in two stages: first we show that there is an ${}^{|R|}\sigma m$-chain from a seed in V_m to c of ψ_m-strength s and then we show that there is no ${}^{|R|}\sigma m$-chain from a seed in V_m to c of ψ_m-strength greater than s.

To show the existence of an ${}^{|R|}\sigma m$-chain from a seed in V_m to c of ψ_m-strength s, we use an inductive argument. If $s = {}^1r = 1$, then the desired result is assured by **A**. Now let $i > 1$ and $s = {}^i r$. Assume that, for $1 \le j < i$, whenever a spel d is such that ${}^{|R|}\sigma_m^d = {}^j r$, then there is an ${}^{|R|}\sigma m$-chain from a seed in V_m to d of ψ_m-strength ${}^j r$.

By **B** there is a chain $\left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle$ of ψ_m-strength s such that $c^{(0)} \in {}^{(i-1)}U$, ${}^{|R|}\sigma_m^{c^{(0)}} > 0$, $c^{(K)} = c$ and, for $1 \le k \le K$, $c^{(k)} \notin {}^{(i-1)}U$. We are now going to show that $\left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle$ is an ${}^{|R|}\sigma m$-chain by showing that, for $1 \le k \le K$, ${}^{|R|}\sigma_m^{c^{(k)}} = s$. Otherwise, consider the smallest $k \ge 1$ that violates this equation. Then we have that ${}^{|R|}\sigma_m^{c^{(k-1)}} \ge s$ and ${}^{|R|}\sigma_m^{c^{(k)}} < s$ (recall that $c^{(k)} \notin {}^{(i-1)}U$). This combined with the fact that $\psi\left(c^{(k-1)}, c^{(k)}\right) \ge s$ violates the consistency of $\left(c^{(k-1)}, c^{(k)}\right)$. Since $c^{(0)} \in {}^{(i-1)}U$ and ${}^{|R|}\sigma_m^{c^{(0)}} > 0$, ${}^{|R|}\sigma_m^{c^{(0)}} = {}^j r$ for some $1 \le j < i$ and, by the induction hypothesis, there is an ${}^{|R|}\sigma m$-chain from a seed in V_m to $c^{(0)}$ of ψ_m-strength ${}^j r > s$. Appending $\left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle$ to this chain we obtain ${}^{|R|}\sigma m$-chain from a seed in V_m to c of ψ_m-strength s.

Now we show that there is no ${}^{|R|}\sigma m$-chain from a seed in V_m to c of ψ_m-strength greater than s. This is clearly so if $s = 1$. Suppose now that $s < 1$ and that $\left\langle c^{(0)}, \cdots, c^{(K)} \right\rangle$ is an ${}^{|R|}\sigma m$-chain from a seed in V_m of ψ_m-strength $t > s$. We now show that, for $0 \le k \le K$, ${}^{|R|}\sigma_m^{c^{(k)}} \ge t$. From this it follows that $c^{(K)}$ cannot be c and we are done. Since $c^{(0)}$ is a seed in V_m, ${}^{|R|}\sigma_m^{c^{(0)}} = 1 \ge t$. For $k > 0$, induction that makes use of the consistency of $\left(c^{(k-1)}, c^{(k)}\right)$ leads to the desired result.

To complete the proof that $\sigma = {}^{|R|}\sigma$ satisfies the property stated in Theorem 1(i), consider (2) and (3). Let $c \in V$ and $1 \le m \le M$. Consider first the case when $\sigma_m^c > 0$. By (12) we get that $s_m^c = \sigma_m^c = \sigma_0^c$. On the other hand, if $\sigma_m^c = 0$, then s_m^c is defined by the second line of (2), with a d for which $\sigma_m^d > 0$. If it were the case that $s_m^c \ge \sigma_0^c$, then by (12) we would have that $\min\left(\sigma_0^d, \psi_m(d, c)\right) \ge \sigma_0^c$. By consistency (see (8) and (9) with c and d interchanged), this can only happen if $\sigma_m^c = \sigma_0^c = 0$ and $\psi_m(d, c) = 0$. So if $s_m^c \ge \sigma_0^c$, then it follows that $s_m^c = 0$. Under all circumstances it therefore follows that if the s_n^c are defined by (2), then (3) is valid. $\square$

Proof of Theorem 1(ii)

Suppose that there are two different M-semisegmentations σ and τ of V having the stated property. We choose a spel c, such that $\sigma^c \neq \tau^c$, but for all $d \in V$ such that $\max(\sigma_0^d, \tau_0^d) > \max(\sigma_0^c, \tau_0^c)$, $\sigma^d = \tau^d$. Without loss of generality, we assume that $\sigma_0^c \geq \tau_0^c$, from which it follows that, for some $m \in \{1, \cdots, M\}$, $\sigma_m^c > \tau_m^c (\geq 0)$ and so, by (3), $\sigma_m^c = s_m^c$ and $c \notin V_m$. This implies that there exists a σm-chain $\langle d^{(0)}, \cdots, d^{(L)} \rangle$ in V of ψ_m-strength not less than $\sigma_m^c (> 0)$ such that $d^{(0)} \in V_m$ and $\psi_m(d^{(L)}, c) \geq \sigma_m^c$. Next we show that $\langle d^{(0)}, \cdots, d^{(L)} \rangle$ is a τm-chain.

We need to show that, for $0 \leq l \leq L$, $\tau_m^{d^{(l)}} > 0$. This is true for 0, since $d^{(0)} \in V_m$. Now assume that it is true for $l-1$ $(1 \leq l \leq L)$. Since $\langle d^{(0)}, \cdots, d^{(l-1)} \rangle$ is a τm-chain in V of ψ_m-strength at least $\sigma_m^c (> 0)$ from an element of V_m, we have that $\mu_{\tau,m,V,V_m}(d^{(l-1)}) \geq \sigma_m^c$. Since we also know that $\psi_m(d^{(l-1)}, d^{(l)}) \geq \sigma_m^c$, we get that $t_m^{d^{(l)}} \geq \sigma_m^c$ (where t is defined for τ as s is defined for σ in (2)). The only way $\tau_m^{d^{(l)}}$ could be 0, is if there were an $n \in \{1, \cdots, M\}$ such that $t_n^{d^{(l)}} > t_m^{d^{(l)}}$. Then $\max(\sigma_0^{d^{(l)}}, \tau_0^{d^{(l)}}) \geq \tau_0^{d^{(l)}} = \tau_n^{d^{(l)}} = t_n^{d^{(l)}} > t_m^{d^{(l)}} \geq \sigma_m^c = \sigma_0^c = \max(\sigma_0^c, \tau_0^c)$. By the choice of c, this would imply that $\sigma^{d^{(l)}} = \tau^{d^{(l)}}$, which cannot be since $\sigma_m^{d^{(l)}} \neq 0$.

From the facts that $\langle d^{(0)}, \cdots, d^{(L)} \rangle$ is a τm-chain of ψ_m-strength not less than σ_m^c and that $\psi_m(d^{(L)}, c) \geq \sigma_m^c$, it follows that $\tau_0^c \geq t_m^c \geq \sigma_m^c = \sigma_0^c \geq \tau_0^c$, implying that all the inequalities are in fact equalities. But then $\sigma_m^c = t_m^c = \tau_m^c$, contradicting $\sigma_m^c > \tau_m^c$ and thereby validating uniqueness. $\square$

Proof of Theorem 1(iii)

We observe that it is a consequence of (3) that, for any spel c, $\sigma_0^c = \max_{1 \leq m \leq M} s_m^c$. Let $\langle c^{(0)}, \cdots, c^{(K)} \rangle$ be a chain of positive ϕ_Ψ-strength from a seed spel to an arbitrary spel c. We now show inductively that, for $0 \leq k \leq K$, $\sigma_0^{c^{(k)}} > 0$. This is clearly so for $k = 0$. Suppose now that it is so for $k-1$. Choose an $m\,(1 \leq m \leq M)$ such that $\sigma_0^{c^{(k-1)}} = \sigma_m^{c^{(k-1)}} = s_m^{c^{(k-1)}}$. Then there is a σm-chain of positive ψ_m-strength from a spel in V_m to $c^{(k-1)}$. Since $\psi_m(c^{(k-1)}, c^{(k)}) > 0$, $\sigma_0^{c^{(k)}} \geq s_m^{c^{(k)}} > 0$. $\square$

3 Algorithm

We claim that the picturesque algorithm described in Section 1 produces an output that is essentially the M-segmentation σ of Theorem 1. However, a most direct implementation of that algorithm would not be computationally efficient: many of the iterative steps would result in no change of the status quo, and even if changes were to take place during an iterative step, resources would be wasted on performing actions that can be avoided by a more carefully designed algorithm that aims at producing the same output.

In [6] we presented a greedy (and hence efficient; see [11]) algorithm for this purpose. It makes use of a priority queue H (a binary heap) of spels c, with associated keys σ_0^c [11]. Such a priority queue has the property that the key of the spel at its head is maximal (its value is denoted by Maximum-Key(H), which is defined to be 0 if H is empty). As the algorithm proceeds, each spel is inserted into H exactly once (using the operation $H \leftarrow H \cup \{c\}$) and is eventually removed from H (using the operation Remove-Max(H), which removes the spel c from the head of the priority queue). At the time when a spel c is removed from H, the vector σ^c has its final value. Spels are removed from H in a non-increasing order of the final value of σ_0^c. We use the variable r to store the current value of Maximum-Key(H). The MOFS (Multi-Object Fuzzy Segmentation) algorithm below shows a detailed specification using the conventions adopted in [11].

We now demonstrate the correctness of this algorithm in the sense that we indicate why it produces the ${}^{|R|}\sigma$ defined by (4) and (6). We do not consider it necessary to give a formal proof here, a discussion of the relationship of the operation of the MOFS algorithm to the definition should suffice.

The process is initialized (Steps 1-9) by first setting σ_m^c to 0, for each spel c and $0 \leq m \leq M$. Then, for every seed spel $c \in V_m$, c is put into H and both σ_0^c and σ_m^c are set to 1. Following this, r is also set to 1. At the end of the initialization, the following conditions are satisfied.

(i) σ is an M-semisegmentation of V.
(ii) A spel c is in H if, and only if, $0 < \sigma_0^c \leq r$.
(iii) $r =$ Maximum-Key(H).
(iv) For $1 \leq m \leq M$, $V_m = \{c \in H \mid \sigma_m^c = r\}$.

It would be nice for easy understanding of the relationship between the algorithm and the definition if σ at this stage were the same as the ${}^1\sigma$ of (4). However, this is not so: in (4) we assign value 1 not only to things in V_m, but also to things that can be reached from V_m by chains of ψ_m-strength 1. It is computationally more efficient to postpone and intermix this action with the next stage. Step 18 of the algorithm is what takes care of this, in a manner that we discuss momentarily.

The initialization is followed by the main loop of the algorithm. At the beginning of each execution of this loop, conditions (i) to (iv) above are satisfied. The main loop is repeatedly performed for decreasing values of r until r becomes 0, at which time the algorithm terminates (Step 10). There are two parts to the main loop, each of which has a very different function.

MOFS algorithm

1. **for** $c \in V$
2.   **do for** $m \leftarrow 0$ **to** M
3.   **do** $\sigma_m^c \leftarrow 0$
4. $H \leftarrow \emptyset$
5. **for** $m \leftarrow 1$ **to** M
6.   **do for** $c \in V_m$
7.   **do if** $\sigma_0^c = 0$ **then** $H \leftarrow H \cup \{c\}$
8. $\sigma_0^c \leftarrow \sigma_m^c \leftarrow 1$
9. $r \leftarrow 1$
10. **while** $r > 0$
11.   **do for** $m \leftarrow 1$ **to** M
12.   **do while** $V_m \neq \emptyset$
13.   **do** remove a spel d from V_m
14. $C \leftarrow \{c \in V \mid \sigma_m^c < \min(r, \psi_m(d,c))$ **and** $\sigma_0^c \leq \min(r, \psi_m(d,c))\}$
15.   **while** $C \neq \emptyset$
16.   **do** remove a spel c from C
17. $t \leftarrow \min(r, \psi_m(d,c))$
18.   **if** $r = t$ **then** $V_m \leftarrow V_m \cup \{c\}$
19.   **if** $\sigma_0^c < t$ **then**
20.   **if** $\sigma_0^c = 0$ **then** $H \leftarrow H \cup \{c\}$
21.   **for** $n \leftarrow 1$ **to** M
22.   **do** $\sigma_n^c \leftarrow 0$
23. $\sigma_0^c \leftarrow \sigma_m^c \leftarrow t$
24.   **while** Maximum-Key$(H) = r$
25.   **do** Remove-Max(H)
26. $r \leftarrow$ Maximum-Key(H)
27.   **for** $m \leftarrow 1$ **to** M
28.   **do** $V_m \leftarrow \{c \in H \mid \sigma_m^c = r\}$

The first part of the main loop (Steps 11-23) is the essential part of the MOFS algorithm. It is in here where we update our best guess so far of the final values of the σ_m^c. A current value is replaced by a larger one if it is found that there is a σm-chain from a seed spel in the initial V_m to c of ψ_m-strength greater than the old value (the previously maximal ψ_m-strength of the known σm-chains of this kind) and it is replaced by 0 if it is found that (for an $n \neq m$) there is a σn-chain from a seed spel in the initial V_n to c of ψ_n-strength greater than the old value of σ_m^c.

To understand the relationship of the main loop of the algorithm to the definition in (6) consider the following. The r in the algorithm corresponds to the ${}^i r$ in the definition. When the loop is entered, the set V_m contains some (but not necessarily all) spels $c \in {}^i U$ for which ${}^i\sigma_m^c > 0$. However, as the execution of the loop proceeds, all spels that satisfy this condition will get put into V_m

(in Step 18).

For the sake of computational efficiency, the algorithm does something that is not directly reflected in definition (6): as soon as an opportunity arises, it greedily estimates values ${}^{j}\sigma_m^c$ for $j \geq i$. Although some of this effort maybe wasted, in the sense that the estimated value will be replaced by another later on, the greedy strategy allows us to avoid having to search explicitly for spels that satisfy the rather complicated condition in the second line of (6).

The purpose of the second part of the main loop (Steps 24-28) is to restore the satisfaction of conditions (iii) and (iv) above for a new (smaller) value of r. It is here that the use of the priority queue structure of H comes into its own: it allows us to skip over steps implied by the inductive definition during which nothing would happen (essentially because we would have ${}^{i}\sigma_m^c =^{(i-1)} \sigma_m^c$).

4 Experiment

To illustrate the use of the MOFS algorithm we segmented an image defined on a V of the type specified in (1) [1]. Fig. 2 shows a 400×397 magnetic resonance image of the head on the left and a 4-segmentation of it on the right.

The way we specify ψ_m and V_m ($1 \leq m \leq 4$) for such an image is the following. We click on some spels in the image to identify them as belonging to the mth object and the V_m is formed by these points and their 8 neighbors. We define g_m to be the mean and h_m to be the standard deviation of the brightness of all spels in V_m and a_m to be the mean and b_m to be standard deviation of the absolute differences of brightness for all adjacent pairs of spels in V_m. Then we define $\psi_m(c,d)$ to be 0 if c and d are not adjacent and to be $[\rho_{g_m,h_m}(g) + \rho_{a_m,b_m}(a)]/2$ if they are, where g is the mean and a is the absolute difference of the brightnesses of c and d and the function $\rho_{r,s}(x)$ is obtained from a Gaussian distribution with mean r and standard deviation s multiplied by a constant so that the peak value becomes 1.

For this segmentation we selected seed points belonging to various anatomically relevant parts (for example, the red seed points were used to identify brain tissue). The segmentation shown on the right of Fig. 2 actually tells us more than just to which object a spel belongs (as indicated by its hue), it also encodes in the brightness of each spel its grade of membership. In fact, one can identify the ventricular cavities inside the brain due to their having low brightness values in the red object.

[1] For examples using images defined on the hexagonal grid and on the face centered cubic grid see [6] and [12], respectively.

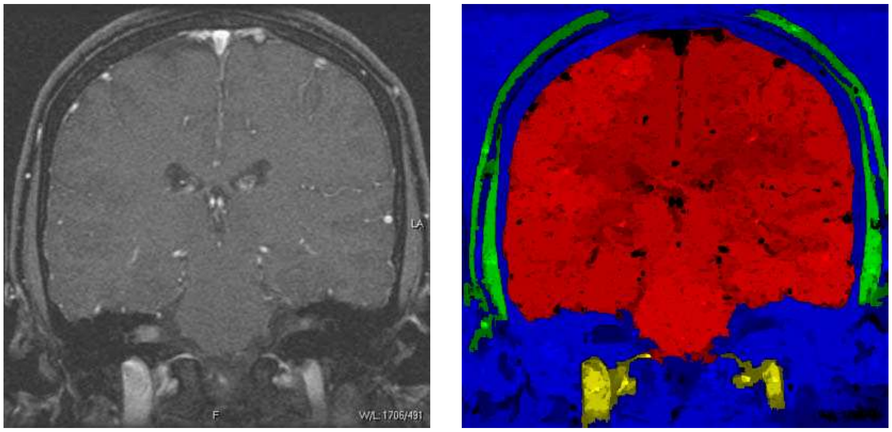

Figure 2. MRI of a patient and a 4-segmentation of it.

The execution time needed by our implementation of the MOFS algorithm to segment the image shown in Fig. 2 was 1.26 seconds using a Xeon$^{\mathrm{TM}}$ 1.7 GHz personal computer, or approximately $8\mu s$ per spel.

5 Comparison with the Approach of Udupa, Saha, and Lotufo

In a recent paper [7] on the topic of segmentation of multiple objects using fuzzy connectedness Udupa, Saha, and Lotufo claimed that the theoretical results of [6], which are represented above, are particular cases of the results described by them. We disagree with this claim, and in this section we present the reasons for our disagreement. These reasons fall into three categories: (1) our approach is more general than that of [7], (2) even in the special cases where both approaches are applicable, they produce different M-segmentations, and (3) the mathematical nature of our main result (Theorem 1) is quite different from anything presented in [7].

As opposed to our general approach, in [7] the only Vs which are discussed are of the form (1) and M is restricted to be 2. The latter is justified on the basis that, for any one of the objects, all the other objects can be considered to be its "background" and so there is no loss of generality. We do not think that this justification is valid in all cases and, even when it is valid, it seems to us desirable to achieve simultaneous M-segmentations of the type illustrated in Fig. 2.

To compare further our approach to that of [7] we need to make precise how objects are defined in [7]. Two different ways of defining objects are presented there.

The first way is called *Relative Fuzzy Connectedness* (*RFC*). For its application it is assumed that $M = 2$, V is some set defined by (1) and $\Psi = (\psi, \psi)$

for some reflexive and symmetric fuzzy spel affinity ψ, and that both sets of seed spels V_1 and V_2 have exactly one element. Under these restrictions, RFC defines a 2-segmentation as follows. For $1 \leq m \leq 2$ and for any $c \in V$, let μ_m^c denote the ψ-strength of the strongest chain (in V) from (the unique element of) V_m to c. Then, let

$$\sigma_1^c = \begin{cases} \mu_1^c, \text{ if } \mu_1^c > \mu_2^c, \\ 0, \quad \text{otherwise}, \end{cases} \tag{13}$$

$$\sigma_2^c = \begin{cases} \mu_2^c, \text{ if } \mu_1^c \leq \mu_2^c, \\ 0, \quad \text{otherwise}, \end{cases} \tag{14}$$

and $\sigma_0^c = max\left\{\sigma_1^c, \sigma_2^c\right\}$. Clearly, σ is a 2-semisegmentation of V. It is not difficult to prove, using the connectedness of V under the fuzzy spel affinity ψ, that σ is a 2-segmentation.

To illustrate this definition, consider the 2-fuzzy graph $\left(\overline{V}, \left(\overline{\psi_1}, \overline{\psi_1}\right)\right)$ with $\overline{V_1} = \{(0)\}$ and $\overline{V_2} = \{(-1)\}$. It is easy to see that the resulting 2-segmentation will be $\sigma^{(-1)} = (1, 0, 1)$, $\sigma^{(0)} = (1, 1, 0)$, and $\sigma^{(1)} = (0.25, 0, 0.25)$, the last due to the fact that $\mu_1^{(1)} = \mu_2^{(1)} = 0.25$.

Even if we restrict ourselves to that subset of M-fuzzy graphs and seed spels to which RFC is applicable, there are several differences between the 2-segmentation produced by Theorem 1 and the one determined by RFC. We now discuss these.

First and foremost, given a 2-fuzzy graph and two sets of seed spels V_1 and V_2, the 2-segmentation produced by Theorem 1 and by RFC can be different. For the example presented in the second paragraph above, Theorem 1 will produce the already discussed 2-segmentation $\overline{\sigma}$, which is essentially different from the σ produced by RFC: in $\overline{\sigma}$ (1) belongs to Object 1 and RFC tells us that it is in the background (Object 2). This is because RFC does not have the concept of the above discussed "blocking" of the chain $\langle(-1), (0), (1)\rangle$ by the seed spel (0) of Object 1. We consider this to be a disadvantage of RFC (but this is more a matter of opinion than a supportable hypothesis).

RFC has a "robustness" property (Proposition 2.4 of [7]) which in our terminology can be restated as follows. If σ is the 2-segmentation defined by RFC and $\sigma_1^q > 0$, then if we replace V_1 by $\widetilde{V_1} = \{q\}$, we get by RFC a 2-segmentation $\widetilde{\sigma}$ such that, for all $c \in V$, $\sigma_1^c > 0$ if, and only if, $\widetilde{\sigma}_1^c > 0$. While it can indeed be argued that this is a desirable property (as it is done in [7]), there are situations where it seems to us to be counterproductive. For example, in the case considered in the previous paragraph (in which $\overline{\sigma}_1^{(0)} = 1$ and $\overline{\sigma}_2^{(0)} = 0$) we had that $\overline{\sigma}_1^{(1)} = 0.25 > 0$. We find this quite acceptable. However, if we replace $\overline{V_1} = \{(0)\}$ by the set $\widetilde{V_1} = \{(-1)\}$, then we get $\widetilde{\sigma}_1^{(0)} = 0$ and $\widetilde{\sigma}_2^{(0)} = 0.5$. This

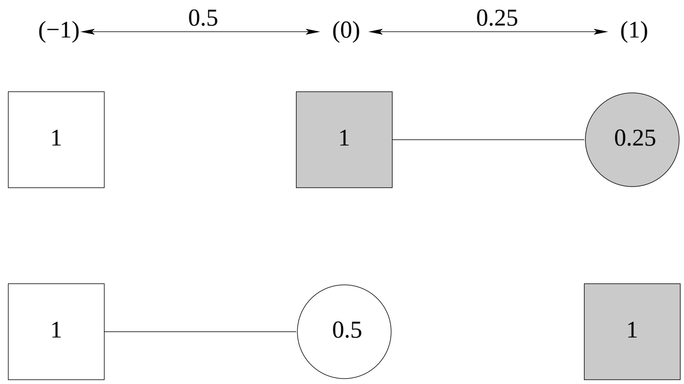

Figure 3. Illustration of a desirable lack of "robustness" in the 2-segmentation determined by Theorem 1. The top row describes the 2-fuzzy graph by specifying the nonzero fuzzy spel affinities. In the other two rows there is shading for the object and no shading for the background, squares for seed spels and circles for other spels, the numbers indicates the grades of membership and the lines indicate the affinity which determines how a (not seed) spel is attached. In the middle row (1) is attached to the object since the path from the background seed spel (-1) is blocked by the object seed spel (0). In the bottom row, (0) is attached to the background, since its seed (-1) has a stronger fuzzy spel affinity to (0) than does the seed (1) of the object.

seems to us quite appropriate, even though it violates the robustness criterion of [7]; see Fig. 3.

Another difference is that the definition in RFC is not symmetric; if we interchange V_1 and V_2 that does not result in interchanging σ_1^c and σ_2^c (see the asymmetry in the definitions (13) and (14)). As a result of this, even though the "object" (Object 1 in our terminology) produced by RFC is guaranteed to be ψ-connected, the "background" (Object 2 in our terminology) is not. This is illustrated in the example $\left(\overline{V},\left(\overline{\psi_1},\overline{\psi_1}\right)\right)$ above, in which the background produced by RFC is disconnected. The M-segmentation defined by Theorem 1 is perfectly symmetric: If we permute the ψ_ms and the V_ms in the same way, then we will get exactly the corresponding permutation of the σ_m^cs (and the connectedness of all spels in an object to at least one seed spel of the object will be preserved). We consider this also a disadvantage of RFC.

To overcome the lack of ability of RFC to achieve some desired results, [7] introduces a second method of object definition: *iterative relative fuzzy connectedness* (*IRFC*). Translated into our terminology, IRFC defines objects as follows.

Given a 2-fuzzy graph $(V,(\psi,\psi))$ and two sets of seed spels V_1 and V_2 (with all the previously stated restrictions in the approach of [7] implied), IRFC produces a sequence ${}^0\psi_2, {}^1\psi_2, \cdots$ of spel-adjacencies and a sequence of ${}^0\sigma, {}^1\sigma, \cdots$ of 2-segmentations defined as follows. ${}^0\psi_2 = \psi$ and ${}^0\sigma$ is the 2-segmentation defined by RFC. Now assume that, for some $i > 0$, we have already obtained

${}^{i-1}\psi_2$ and ${}^{i-1}\sigma$. For all $c, d \in V$, we define

$$
{}^{i}\psi_2(c,d) = \begin{cases} 1, & \text{if } c = d, \\ 0, & {}^{i-1}\sigma_1^c > 0 \text{ or } {}^{i-1}\sigma_1^d > 0, \\ \psi(c,d) & \text{otherwise.} \end{cases} \tag{15}
$$

Using the notation ${}^{i}\psi_1 = \psi$, for all i, we define, for $1 \le m \le 2$ and for any $c \in V$, ${}^{i}\mu_m^c$ as the ${}^{i}\psi_m$-strength of the strongest chain (in V) from V_m to c. Then ${}^{i}\sigma$ is defined just as σ is defined in RFC using (13) and (14), but with μ_m^c replaced by ${}^{i}\mu_m^c$ everywhere. Note that the middle line of (15) causes the "blocking" of chains in Object 2 by spels in Object 1. Note also that the already indicated asymmetry in RFC between object and background seems to get even worse in the definition of IRFC.

To illustrate this definition, consider the same 2-fuzzy graph $\left(\overline{V}, \left(\overline{\psi_1}, \overline{\psi_1}\right)\right)$ and seed spels that we used to illustrate RFC. Accordingly ${}^{0}\sigma^{(-1)} = (1,0,1)$, ${}^{0}\sigma^{(0)} = (1,1,0)$ and ${}^{0}\sigma^{(1)} = (0.25, 0, 0.25)$. Using (15), we see that ${}^{1}\psi_2(c,d) = 0$ if $c = (0)$ or $d = (0)$. This causes the value of ${}^{1}\mu_2^{(1)}$ to be 0 (while ${}^{1}\mu_1^{(1)} = 0.25$), and so ${}^{1}\sigma^{(1)} = (0.25, 0.25, 0)$; i.e. (1) gets assigned to Object 1 rather than to Object 2. Further iterations will not change the 2-segmentation; i.e., ${}^{i}\sigma = {}^{1}\sigma$, for $i \ge 1$. Note that in fact this ${}^{1}\sigma$ is the very $\overline{\sigma}$ that is determined by Theorem 1 for the same 2-fuzzy graph and seed spels.

However, this is not always the case; we now give an example in which the 2-segmentation determined by Theorem 1 is different from all the 2-segmentations produced by IRFC. Consider the 2-fuzzy graph $\left(\overline{V}, (\psi, \psi)\right)$, where ψ is determined by $\psi((-1),(0)) = 1$, $\psi((0),(1)) = 0.5$, and $\psi((-1),(1)) = 0$, and $\overline{V_1} = \{(0)\}$ and $\overline{V_2} = \{(-1)\}$. The 2-segmentation determined by Theorem 1 is $\sigma^{(-1)} = (1,1,1)$, $\sigma^{(0)} = (1,1,1)$ and $\sigma^{(1)} = (0.5, 0.5, 0.5)$. On the other hand, it is easy to see that, for all $i \ge 0$, the 2-segmentation provided by IRFC is ${}^{i}\sigma^{(-1)} = (1,0,1)$, ${}^{i}\sigma^{(0)} = (1,0,1)$ and ${}^{i}\sigma^{(1)} = (0.5, 0, 0.5)$. That is, Theorem 1 provides us with a 2-segmentation in which every spel belongs to both objects, but IRFC provides us with a 2-segmentation in which every spel is only in Object 2 (the background), including even the seed spel of Object 1. This seems to us a disadvantage of IRFC. This disadvantage becomes even more obvious if we do something that is allowed in [6] but not in the theory of [7], namely if we replace the 2-fuzzy graph by $\left(\overline{V}, (\psi_1, \psi_2)\right)$ where ψ_1 is the ψ defined above and ψ_2 is also ψ except for $\psi_2((0),(1)) = 0.25$. Then, in our opinion quite reasonably, the 2-segmentation provided by Theorem 1 is $\sigma^{(-1)} = \sigma^{(0)} = (1,1,1)$ and $\sigma^{(1)} = (0.5, 0.5, 0)$; i.e., (1) belongs only to Object 1. The ability to achieve this depends on our freedom of selecting ψ_1 and ψ_2 independently and it cannot be imitated by IRFC as defined in [7]. (There is an alternative version of IRFC in which the condition in the middle row of

(15) is replaced by: if ${}^{i-1}\sigma_2^c = 0$ or ${}^{i-1}\sigma_2^d = 0$. However, that version generates exactly the same 2-segmentation for the example provided in this paragraph as the version of (15).)

Finally we comment on Theorem 1. Its general nature is the following. “Let G be a connected graph. A partial labeling of the nodes of G is said to have Property X if the label at each node can be determined from the labels assigned to the other nodes by a Procedure Y. We claim that for every graph G, there is one and only one partial labeling that has Property X, and it is in fact a total labeling.” This is a result of some substance: there is no *a priori* reason to believe that, for all connected graphs, there would necessarily be a labeling with Property X, or that that labeling (if it exists) should be unique and/or total. While it is true that RFC and IRFC produce unique labellings, that fact is, mathematically speaking, much weaker than our result: RFC and IRFC are deterministic algorithms, and so as long as they terminate they will of course have a unique output. It is much more difficult to prove that a labeling defined by an inherent property exists and is unique than to prove that a labeling produced by an algorithm exists and is unique.

6 Discussion

The main purpose of this paper was to present material associated with our MOFS algorithm for simultaneous fuzzy segmentation of multiple objects that complements that presented in [6]. In particular, we proved that an M-semisegmentation satisfying the condition in Theorem 1(i) exists and indicated why the MOFS algorithm does indeed produce the M-segmentation of Theorem 1.

We also compared our method with the method of [7], commenting on the greater generality of our method, on the differences between the methods, and on the theoretical importance of Theorem 1.

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References

[1] B.M. Carvalho, C.J. Gau, G.T. Herman, and T.Y. Kong. Algorithms for fuzzy segmentation. *Pattern Anal. Appl.*, 2:73–81, 1999.

[2] H.A. Moghaddam and J.F. Lerallut. Volume visualization of the heart using MRI 4D cardiac images. *J. Comput. Inform. Tech.*, 6:215–228, 1998.

[3] J.K. Udupa and S. Samarasekera. Fuzzy connectedness and object definition: Theory, algorithms and applications in image segmentation. *Graph. Models Image Proc.*, 58:246–261, 1996.

[4] A. Rosenfeld. Fuzzy digital topology. *Inform. and Control*, 40:76–87, 1979.

[5] G.T. Herman. *Geometry of Digital Spaces*. Birkhäuser, Boston, MA, 1998.

[6] G.T. Herman and B.M. Carvalho. Multiseeded segmentation using fuzzy conectedness. *IEEE Trans. on Pattern Anal. Mach. Intell.*, 23:460–474, 2001.

[7] J.K. Udupa, P.K. Saha, and R.A. Lotufo. Relative fuzzy connectedness and object definition: Theory, algorithms, and applications in image segmentation. *IEEE Trans. on Pattern Anal. Mach. Intell.*, 24:1485 –1500, 2002.

[8] C.T. Zahn. Graph-theoretic methods for detecting and describing Gestalt clusters. *IEEE Trans. Comp.*, 1:68–86, 1971.

[9] R.O. Duda and P.E. Hart. *Pattern Classification and Scene Analysis*. John Wiley & Sons, New York, 1973.

[10] A.K. Jain, M.N. Murty, and P.J. Flynn. Data clustering: a review. *ACM Comput. Surveys*, 31:264–323, 1999.

[11] T.H. Cormen, C.E. Leiserson, and R.L. Rivest. *Introduction to Algorithms*. MIT Press, Cambridge, MA, 1990.

[12] B.M. Carvalho, E. Garduño, and G.T. Herman. Multiseeded fuzzy segmentation on the face centered cubic grid. In S. Singh, N. Murshed, and W. Kropatsch, editors, *ICAPR 2001 (LNCS 2013)*, pages 339–348, Springer-Verlag, Berlin, 2001.